Advanced Algorithms

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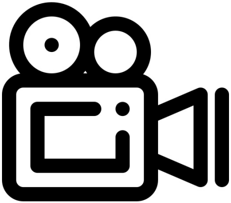
Edmonds-Gallai Decomposition

Lecture 3

Recording of this Lecture

This lecture will be recorded

- Recording only of the lecturers by themselves.
- If there are questions from the audience, please make a clear signal if the microphone shall be muted.
- Our goal is to record the lecture, but it is no guarantee that each lecture will be recorded.



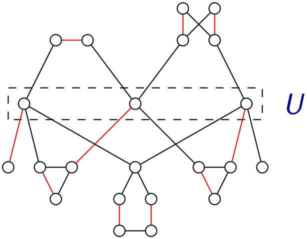
Edmonds-Gallai Decomposition

Tutte-Berge Formula

Question

What is the structure of maximum-matchings?
Can we find all matchings efficiently?

- $\nu(G)$ denotes the cardinality of a maximum matching in G
- For $U \subseteq V$, let $G - U$ denote the subgraph of G obtained by deleting the vertices of U and all edges incident with them.
- $o(G - U)$ denotes the number of connected components of $G - U$ that contain an odd number of vertices.



Tutte-Berge Formula

- At least one unmatched vertex v in an odd component, since any matching necessarily covers an even number of vertices.
- It is possible that we could increase the size of M by matching v with some vertex in U.
- We can add at most $|U|$ edges to M in this manner, since the vertices in U will eventually all be matched
- This shows that the maximum size of a matching is upper-bounded by $(|V| + |U| - o(G - U))/2$, for any subset U.

Theorem (Tutte-Berge Formula)

Let $G = (V, E)$ be a graph. Then

$$\nu(G) = \max_M |M| = \min_{U \subset V} (|V| + |U| - o(G - U))/2 ,$$

where the maximization is over all matching M in G.

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Edmonds-Gallai Decomposition

Theorem (Edmonds-Gallai Decomposition)

Given a graph G, let

$$D(G) := \{v : \text{there exists a maximum size matching missing } v\} ,$$
$$A(G) := \{v : v \text{is a neighbor of some } u \in D(G) \text{ but } v \notin D(G)\} ,$$
$$C(G) := V(G) \setminus (D(G) \cup A(G)) .$$

Then:

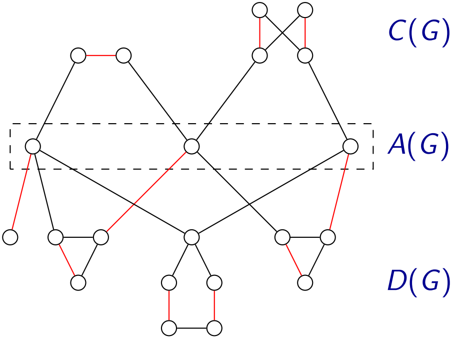
(i) $U = A(G)$ achieves the minimum on the right side of the Tutte-Berge formula,

(ii) $C(G)$ is the union of the even-sized components of $G \setminus A(G)$,

(iii) $D(G)$ is the union of the odd-sized components of $G \setminus A(G)$,

(iv) Every odd-sized component of $G \setminus A(G)$ is factor-critical. (A graph H is factor-critical if for every vertex v, there is a matching in H whose only unmatched vertex is v.)

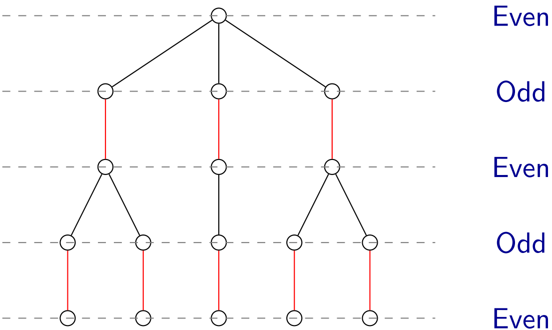
Finding the Edmonds-Gallai Decomposition



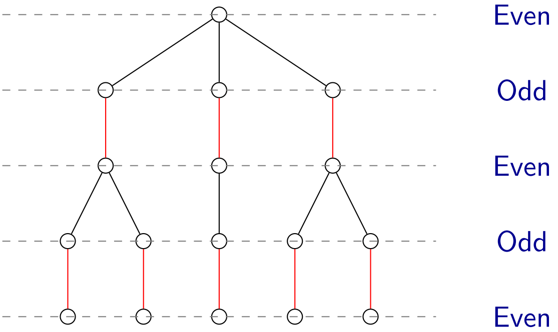
- The Edmonds-Gallai Decomposition can be found using Edmonds' Matching Algorithm.
- But how exactly?

Finding the Edmonds-Gallai Decomposition

- We first compute a maximum matching M in G.
- Let X be the set of vertices not matched by M.
- Consider all the vertices that can be reached by an alternating path from $x \in X$.



Finding the Edmonds-Gallai Decomposition



- Motivated by this, we define the following three subsets of $V(G)$:

  $\text{Even} := \{v : \exists \text{ an alternating path of even length from } X \text{ to } v\}$,

  $\text{Odd} := \{v : \exists \text{ an alternating path from } X \text{ to } v\} \setminus \text{Even}$,

  $\text{Free} := \{v : \nexists \text{ an alternating path from } X \text{ to } v\}$.

Claims for proving the Edmonds-Gallai Decomposition

- We now prove a few claims that help us proving the Theorem.

Claim 1

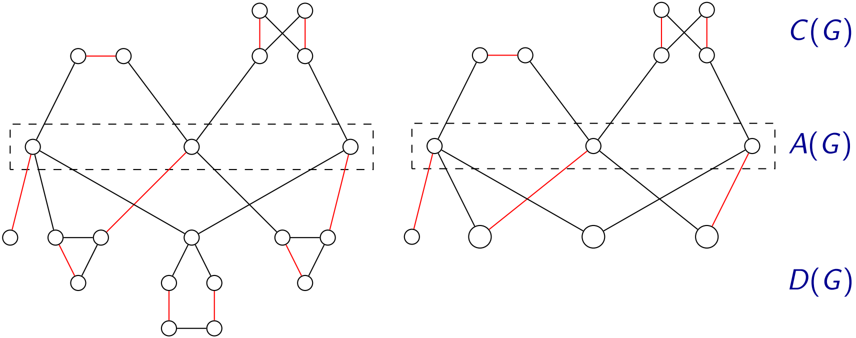
If there is an edge from a vertex $u \in \text{Even}$ to some v, then there is an alternating walk of odd length from X to v, and there is an alternating path from X to v.

Corollary 1

In G there is no edge between Even and Free.

The shrunk Graph

- We define the shrunk graph G_k to be the graph obtained in the final iteration of the execution of Edmonds' algorithm on G (after shrinking all blossoms).



The shrunk Graph

- We define the shrunk graph G_k to be the graph obtained in the final iteration of the execution of Edmonds' algorithm on G (after shrinking all blossoms).
- Let M_k be the maximum size matching in G_k computed by the algorithm - M_k is just the matching M from which the edges of the blossoms shrunk in G_k have been removed.
- Note that the set of the vertices of G_k that are unmatched in M_k is still X.
- All vertices of a blossom become even whenever we expand them.

Claims for proving the Edmonds-Gallai Decomposition

- We prove a few more claims that help us proving the Theorem.

Claim 2

In G_k there is no edge between two even vertices.

Claim 3

Even $= D(G) = \{v : \exists$ a maximum-size matching missing $v\}$.

Claim 4

Odd $= A(G) = \{v : v$ is a neighbor of some $u \in D(G),$ but $v \notin D(G)\}$.

Claim 5

Free $= C(G) = V(G) \setminus (D(G) \cup A(G))$.

Claims for proving the Edmonds-Gallai Decomposition

- We prove a few more claims that help us proving the Theorem.

Claim 6

$|M \cap C(G)| = |C(G)|/2$.

Claim 7

For every connected component H of $G \setminus A(G) \cap D(G)$:

(a) either $|X \cap H| = 1$ and $|M \cap \delta(H)| = 0$; or $|X \cap H| = 0$ and $|M \cap \delta(H)| = 1$, where $\delta(H)$ is the set of edges with exactly one endpoint in H.

(b) H is factor-critical.

Claim 8

$|M| = \frac{1}{2}\Big(|V| + |A(G)| - o(G \setminus A(G))\Big)$.

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Recap and Outlook

- Edmonds-Gallai Decomposition
  - Tutte-Berge Formula
  - Strucutre of all maximum matchings
  - Finding U (for Tutte-Berge Formula) and the Edmonds-Gallai Decomposition
- Next Lectures:
  - Stable matchings