Advanced Algorithms

Nicole Megow (Universität Bremen)

SoSe 2025

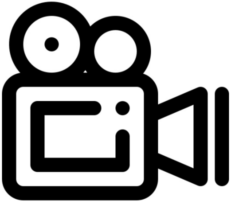
Network Flows: Min-Cost Flows

Lecture 8

Recording of this Lecture

This lecture will be recorded

- Recording only of the lecturers by themselves.
- If there are questions from the audience, please make a clear signal if the microphone shall be muted.
- Our goal is to record the lecture, but it is no guarantee that each lecture will be recorded.



Networks with Supplies, Demands, and Costs

Definition (Network with costs)

A tuple $\mathcal{N} = (V, E, c, b, p)$ is a network with cost if

- V is a finite set of vertices,
- $E \subseteq V \times V$ is a set of directed edges,
- $c : E \to \mathbb{R}_{\geq 0}$ capacities,
- $b : V \to \mathbb{R}$ is a supply function, and
- $p : E \to \mathbb{R}_{\geq 0}$ is a cost function.

Feasible Flows with Costs

Definition (Feasible flow)

A feasible flow in a network $\mathcal{N} = (V, E, c, b, p)$ is a function $f : E \to \mathbb{R}_{\geq 0}$ such that

- $f(e) \leq c(e)$ for all $e \in E$,
- $\sum_{e\in\delta^+(v)} f(e) - \sum_{e\in\delta^-(v)} f(e) = b(v)$ for all $v \in V$.

Definition (Cost of a flow)

The cost of a flow f in a network $\mathcal{N} = (V, E, c, b, p)$ is defined as

$$\text{cost}(f) = \sum_{e\in E} p(e)f(e).$$

In the min-cost flow problem, we want to find a feasible flow f with minimum cost.

Circulations and Cycles

Definition (Circulation)

A circulation f in a network $\mathcal{N} = (V, E, c, b, p)$ is flow f that is feasible for the network $(V, E, c, 0, p)$.

Lemma

Let f and f' be two feasible flows in a network $\mathcal{N} = (V, E, c, b, p)$. Then, $f' - f$ is a circulation in $\mathcal{N}_f$.

Lemma

Let f be a circulation in a network $\mathcal{N} = (V, E, c, b, p)$. Then, there flows $f_1, \ldots, f_k$ with $k \leq m$ such that $f = f_1 + \ldots + f_k$

- *f_i is a feasible flow in $\mathcal{N}$ for all $i \in [k]$, and*
- *f_i takes positive values only on edges of a cycle C_i in $\mathcal{N}$ for all $i \in [k]$.*

Optimality of Minimum-Cost Flows

Cost of a cycle $\mathrm{cost}(C) = \sum_{e \in C} p(e)$.

A cycle C is negative if $\mathrm{cost}(C) < 0$.

Lemma

A feasible flow f in a network $\mathcal{N} = (V, E, c, b, p)$ is optimal if and only if there is no negative cycle in $\mathcal{N}_f$.

The Generic Cycle-Canceling Algorithm

Cycle-Canceling Algorithm

Input: A network $\mathcal{N} = (V, E, c, b, p)$
Output: An min-cost flow f in $\mathcal{N}$
Let f be any feasible flow in $\mathcal{N}$
while there is a negative cycle C in $\mathcal{N}_f$ **do**
  Augment f along C
return f

Theorem

Let $\mathcal{N} = (V, E, c, b, p)$ be a network with costs. If the values $c(e)$, $b(v)$, and $p(e)$ are integers for all $e \in E$ and $v \in V$, then the generic Cycle-Augmenting algorithm computes a min-cost flow in $\mathcal{N}$ in time $O(m^2 n \cdot C \cdot P)$, where n is the number of vertices, m is the number of edges, C is the maximum capacity of an edge in $\mathcal{N}$, and P is the maximum cost of an edge in $\mathcal{N}$.

The Minimum-Mean Cycle Algorithm

Minimum-Mean Cycle Algorithm

Input: A network $\mathcal{N} = (V, E, c, b, p)$
Output: An min-cost flow f in $\mathcal{N}$

```
Let f be any feasible flow in N
while there is a negative cycle in N_f do
    Let C be a negative cycle in N_f of minimum-mean cost
    Augment f along C
return f
```

Theorem

The Minimum-Mean Cycle-Augmenting Algorithm computes a min-cost flow in an integral network with costs with n vertices and m edges in time $O(n^2 m^3 \log n)$.