Advanced Algorithms

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SoSe 2025

Introduction to Linear Programming

Lecture 10

So far in this course

We have focused on exploiting problem structure to design tailored algorithms for specific combinatorial optimization problems:

- Maximum (weighted) matching
- Maximum s-t-flow and minimum s-t-cut
- Minimum-cost flows
- Scheduling

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$\longrightarrow$ Linear Programming

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- Arbitrary linear inequalities with $\geq, \leq$ and $=$.
- Objective: maximum or minimum.

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Typical LP:

$$
\begin{array}{rl}
\min & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{s.t.} & a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n \geq b_1 \\
& a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n \geq b_2 \\
& \ldots \\
& a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n \geq b_n \\
& \forall i\colon x_i \geq 0, x_i \in \mathbb{R}
\end{array}
$$

Compact form:

$$
\begin{array}{rl}
\min & c^T x \\
\text{s.t.} & Ax \geq b \\
& x \geq 0
\end{array}
$$

First Example: Pizza and Lasagne

Ingredients:

	Pizza	Lasagne	available
Tomatoes	2	3	18
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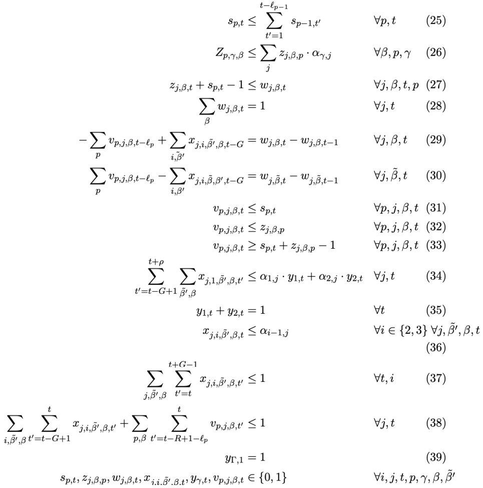
LPs can be arbitrarily complex



$$s_{p,t} \leq \sum_{t'=1}^{t-\ell_{p-1}} s_{p-1,t'} \qquad \forall p, t \qquad (25)$$

$$Z_{p,\gamma,\beta} \leq \sum_{j} z_{j,\beta,p} \cdot \alpha_{\gamma,j} \qquad \forall \beta, p, \gamma \qquad (26)$$

$$z_{j,\beta,t} + s_{p,t} - 1 \leq w_{j,\beta,t} \qquad \forall j, \beta, t, p \qquad (27)$$

$$\sum_{\beta} w_{j,\beta,t} = 1 \qquad \forall j, t \qquad (28)$$

$$-\sum_{p} v_{p,j,\beta,t-\ell_p} + \sum_{i,\tilde{\beta}'} x_{j,i,\tilde{\beta}',\beta,t-G} = w_{j,\beta,t} - w_{j,\beta,t-1} \qquad \forall j, \beta, t \qquad (29)$$

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$$v_{p,j,\beta,t} \leq s_{p,t} \qquad \forall p, j, \beta, t \qquad (31)$$

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$$\sum_{t'=t-G+1}^{t+\rho} \sum_{\tilde{\beta}',\beta} x_{j,1,\tilde{\beta}',\beta,t'} \leq \alpha_{1,j} \cdot y_{1,t} + \alpha_{2,j} \cdot y_{2,t} \qquad \forall j, t \qquad (34)$$

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$$\sum_{i,\tilde{\beta}',\beta} \sum_{t'=t-G+1}^{t} x_{j,i,\tilde{\beta}',\beta,t'} + \sum_{p,\beta} \sum_{t'=t-R+1-\ell_p}^{t} v_{p,j,\beta,t'} \leq 1 \qquad \forall j, t \qquad (38)$$

$$y_{\Gamma,1} = 1 \qquad (39)$$

$$s_{p,t}, z_{j,\beta,p}, w_{j,\beta,t}, x_{i.i.\tilde{\beta}'.\beta.t}, y_{\gamma,t}, v_{p,j,\beta,t} \in \{0,1\} \qquad \forall i, j, t, p, \gamma, \beta, \tilde{\beta}'$$

Verifying feasibility of a given solution is easy, but proving optimality is much harder.

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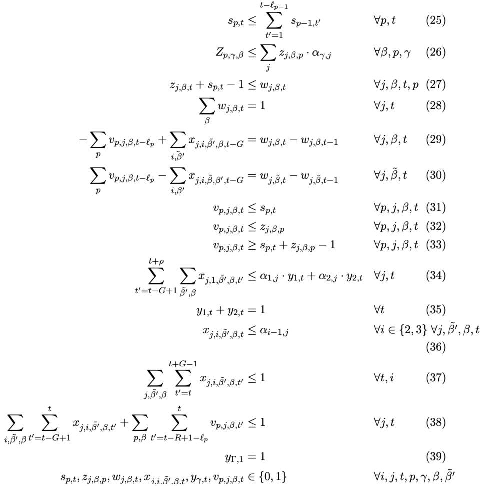

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Verifying feasibility of a given solution is easy, but proving optimality is much harder.

There is good news...!

Fundamental Result: Solving LPs is in P

Theorem

An optimal vector x^* to an LP can be computed in polynomial time with respect to the LP encoding size.

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  - still useful, as we will see

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Theorem

It is NP-complete to decide whether a given integer linear program has a feasible solution.

Roadmap for today

1. Some modeling examples
2. Some background, geometric interpretation and important results
3. Usefulness of LP solutions as a benchmark

   → **LP relaxations**

Modeling Examples

Maximum Matching

Problem: Maximum Matching

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ILP formulation

$$\begin{aligned} \max \sum_{e \in E} x_e & & \\ \text{s.t.} \sum_{e \in \delta(v)} x_e \le 1, & \qquad \text{for all } v \in V \\ x_e \in \{0, 1\}, & \qquad \text{for all } e \in E. \end{aligned}$$

Minimum-Cost Flow

Problem: Minimum-Cost Flow

Given a network $\mathcal{N} = (V, E, c, b, p)$ consisting of a directed graph (V, E) with edge capacities $c_e \in \mathbb{R}_{\geq 0}$, costs $p_e \in \mathbb{R}_{\geq 0}$, and a supply/demand function $b : V \to \mathbb{R}$, find a feasible flow that satisfies all supply to demand and minimizes the total cost.

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$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e & & \\ \text{s.t.} \quad & \sum_{e \in \delta^+(v)} x_e - \sum_{e \in \delta^-(v)} x_e = b(v), & & \text{for all } v \in V \\ & x_e \leq c_e, & & \text{for all } e \in E \\ & 0 \leq x_e, & & \text{for all } e \in E \end{aligned}$$

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The Knapsack Problem

Problem: (Unbounded) Knapsack Problem

There is given a capacity K and an unlimited supply of items of n different types. An item of type i has weight w_i and profit v_i.

Task: Select a subset of items of maximum total profit such that the total weight does not exceed the capacity bound K.

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ILP formulation:

$$\begin{aligned} \max \quad & v_1 \cdot x_1 + v_2 \cdot x_2 + \ldots + v_n \cdot x_n \\ s.t. \quad & w_1 \cdot x_1 + w_2 \cdot x_2 + \ldots + w_n \cdot x_n \leq K \\ & x_i \geq 0 \qquad i \in \{1, \ldots, n\} \\ & x_i \in \mathbb{Z} \qquad i \in \{1, \ldots, n\} \end{aligned}$$

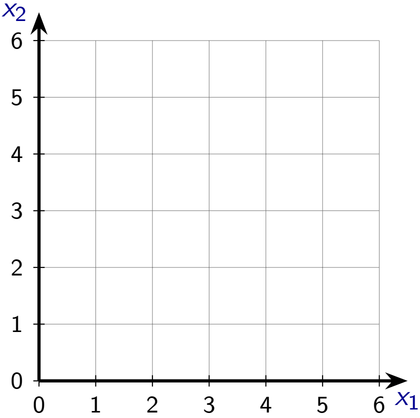
Linear Programming

A Geometrical View

The Real, *n*-dimensional Space

$\mathbb{R}^n := \{x = (x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R}\}$

Elements $x \in \mathbb{R}^n$ can be seen as



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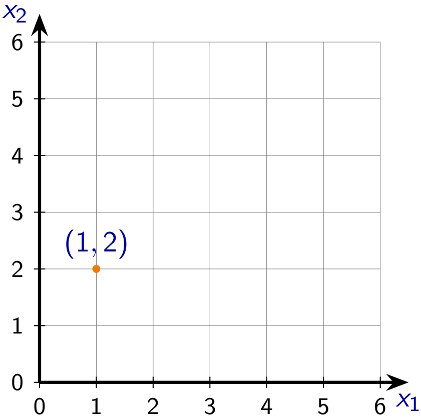

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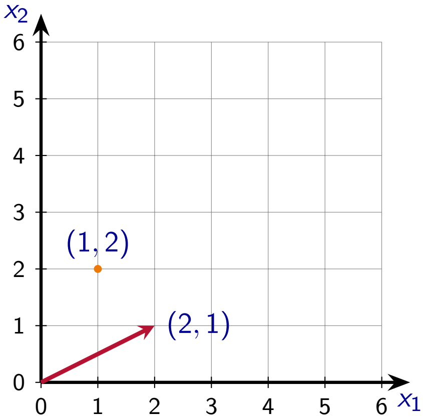

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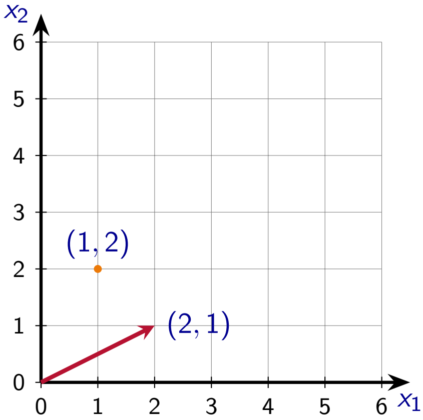
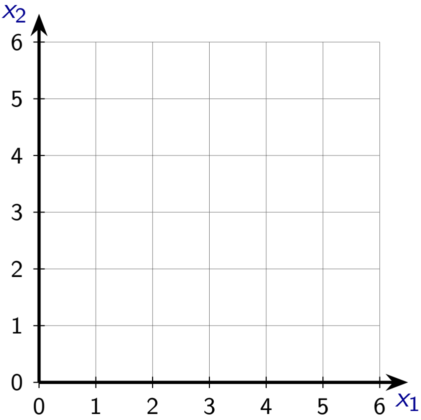
An n-tuple may represent the net profit of n different goods, or their inventory level, or the cost of production, etc.



Linear Equations

Let $a \in \mathbb{R}^n$ be a profit vector.

If you want the profit to be exactly b, how much should you produce?



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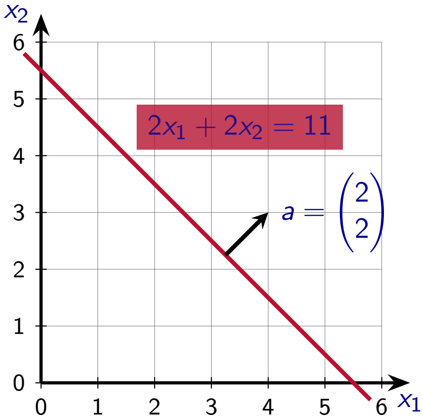
This is a hyperplane in $\mathbb{R}^n$.



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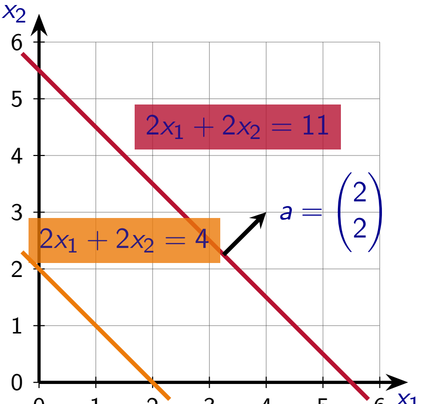
Changing the right hand side b corresponds to "moving" the hyperplane along the vector a.



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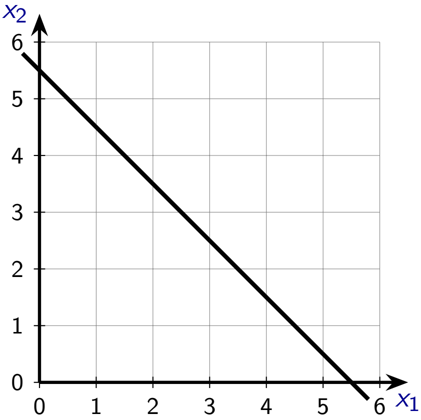

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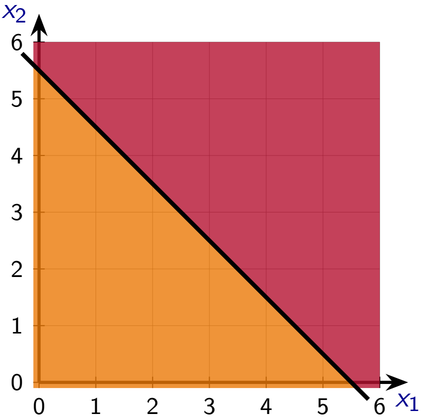
This is a halfspace in $\mathbb{R}^n$.



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$x_2 =$ Number of lasagnas

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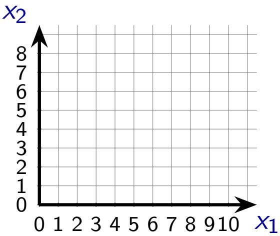

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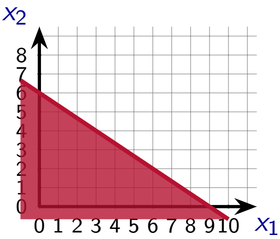

Which points satisfy $2x_1 + 3x_2 \leq 18$?

- Points on $2x_1 + 3x_2 = 18$.
- As well as points below

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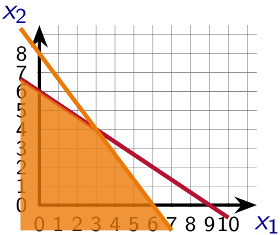

Which points satisfy $4x_1 + 3x_2 \leq 24$?

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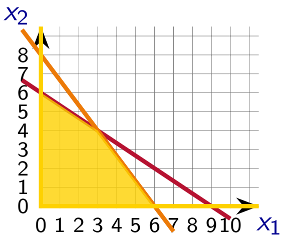

Which points satisfy $x_1, x_2 \geq 0$?

- Points on $x_1 = 0$ or $x_2 = 0$.
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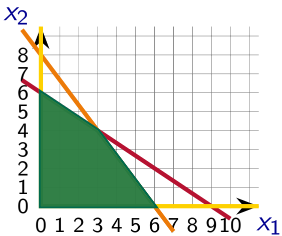

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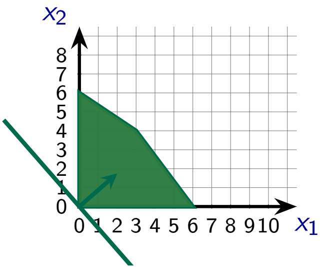

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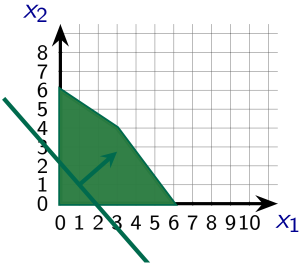

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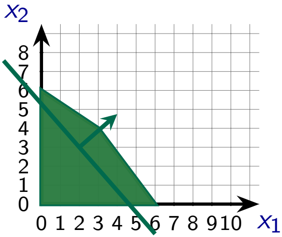

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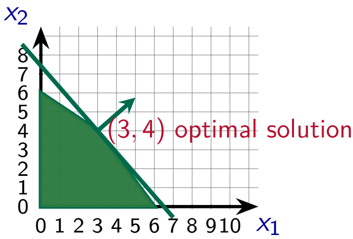

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5. Compute optimal objective function value $z^* = z(x_1^*, x_2^*)$.

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LP Model

x_1 = number of produced pizzas

x_2 = number of produced lasagne

$$
\begin{array}{llll}
\max & 8x_1 + 7x_2 & & \text{(profit)} \\
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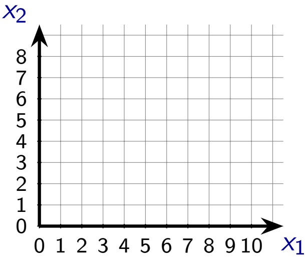
Graphical representation and solution



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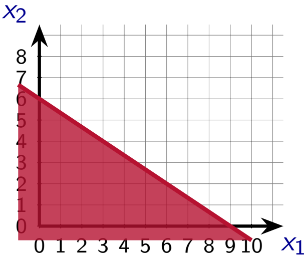

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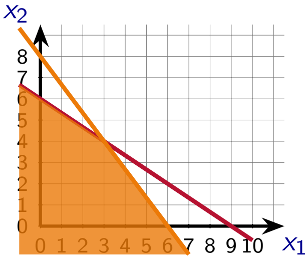

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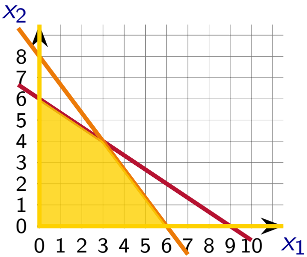

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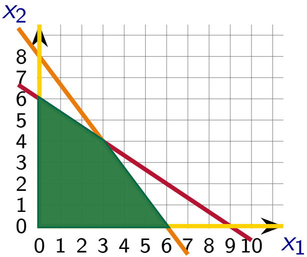

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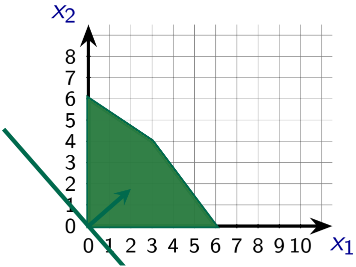

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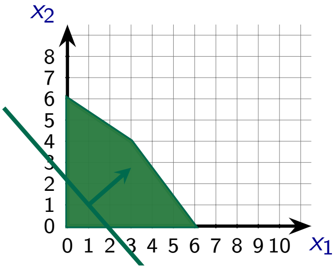

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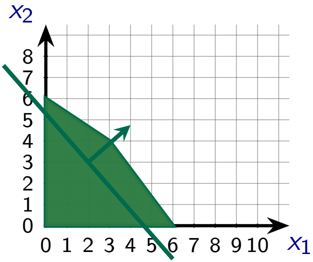

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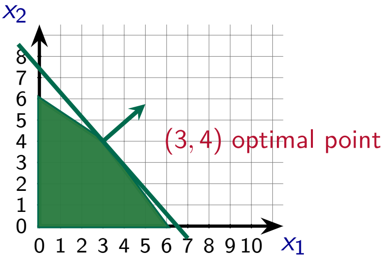


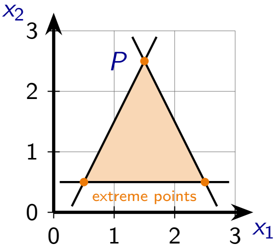
Extreme points

Definition

Let $M \neq \emptyset$ be a convex set. A point $x \in M$ is an extreme point of M if we cannot find two points $y, z \in M \setminus \{x\}$ and a scalar $\lambda \in (0,1)$ such that

$$x = \lambda y + (1-\lambda) z.$$

$$P = \left\{ x \in \mathbb{R}^2 : \begin{array}{rl} x_2 \geq & \frac{1}{2} \\ 2x_1 + x_2 \leq & \frac{11}{2} \\ -2x_1 + x_2 \leq & -\frac{1}{2} \end{array} \right\}$$



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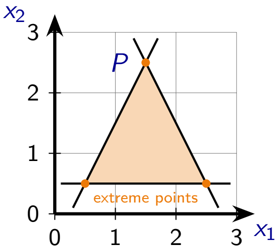

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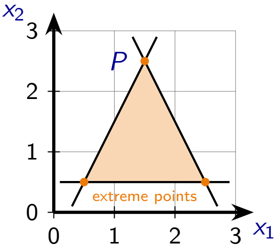

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- Examples of convex sets with infinitely many extreme points: circle or ball

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Definition (equivalent characterization)

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- The set $\{x \in \mathbb{R}^n \,|\, Ax \leq (\geq) b\}$ is called polyhedron.
- The set of feasible solutions of an LP is a polyhedron.

Optimal Solutions

Let $P = P(A, b) = \{x \in \mathbb{R}^n \mid Ax \leq b\} \neq \emptyset$ a polyhedron.

Theorem

If the optimization problem $\min\{c^T x \mid x \in P\}$ has an optimal solution, then at least one optimal solution is attained at a vertex of P.

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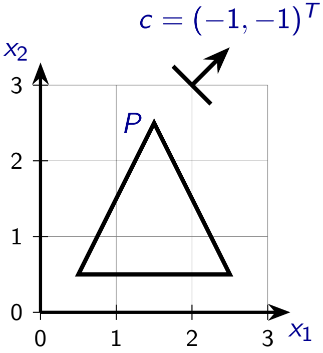

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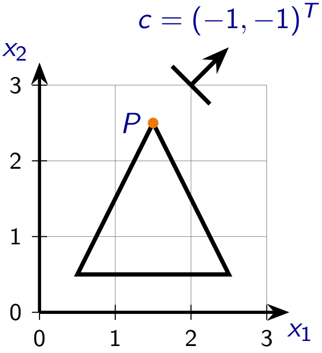
optimal solution



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Example: Unit cube $P = \{x \in \mathbb{R}^n \,|\, 0 \leq x_i \leq 1, i = 1, \ldots, n\}$

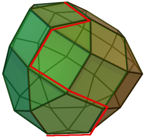
Number of inequalities: $m = 2n$
Number of vertices: 2^n

Enumeration is no option! Smarter algorithms needed.

Solving LPs (First Algorithm)

Simplex method (George Dantzig, 1947)

- First general method for solving LPs
- Starts from some vertex and iteratively moves to an adjacent vertex with better objective function value.
- In general, it does not run in polynomial time.
- However, works very well and fast in practice.



Solving LPs in Polynomial Time

Ellipsoid method (Leonid Khachiyan, 1979)

- First algorithm that solves LPs in polynomial time.
- Builds in earlier work by Shor, Yudin, and Nemirovskii
- Intuitive idea:
  - Binary search for optimal objective value z.
  - Add constraint for target value $c^T x \leq z$.
  - Resulting problem: decide whether a given polytope P is non-empty.
  - Generates sequence of ellipsoids of decreasing volume containing P.
  - For each ellipsoid: verify whether center point is in P (done) or find a separating hyperplane. (separation problem)
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Interior point methods

- By now several poly-time interior point methods are known.

Equivalence of Separation and Optimization

Separation problem: Given a point x and a polyhedron P, decide whether $x \in P$. If $c \notin P$ then give a certificate, i.e., a separating hyperplane (an inequality satisfied by all points in P but not by x).

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Theorem (informally)

Solving the separation problem in polynomial time is equivalent to solving the optimization problem in polynomial time.

Very powerful theorem! (By the Ellipsoid method.)

Example: Minimum Cut

Given $G = (V, E)$ and $s, t \in V$, let $\mathcal{P}$ be the collection of all s-t-paths in G.

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- This is: Find min-weight s-t-path in G with weights $\bar{x}$. (poly-time)

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LPs can be solved in polynomial time!

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- Ellipsoid method

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Optimization = Separation

In general, ILPs cannot be solved in polynomial time, unless P=NP!

Integer Linear Programs

Integer Linear Programs (ILP)

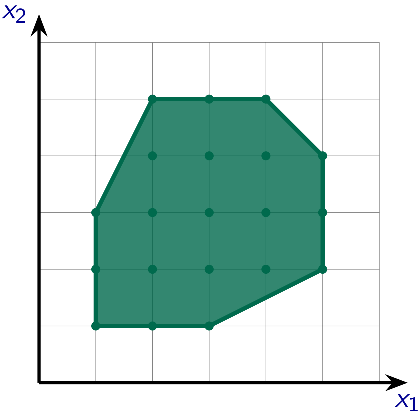
For many problems we find LP formulations only with integrality constraints. → Solving ILPs is NP-hard (in general).

Example:

$$\max \sum_{i=1}^{n} v_i \cdot x_i$$

$$s.t. \sum_{i=1}^{n} w_i \cdot x_i \leq K$$

$$x_i \in \mathbb{Z}_+ \qquad i \in \{1, \ldots, n\}$$



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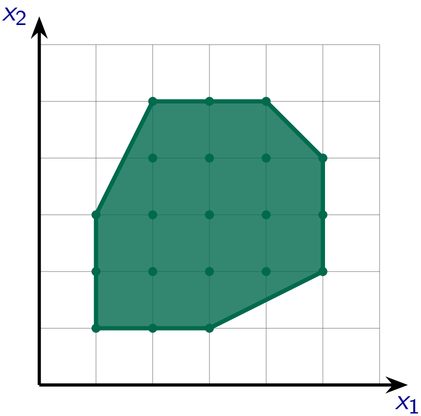

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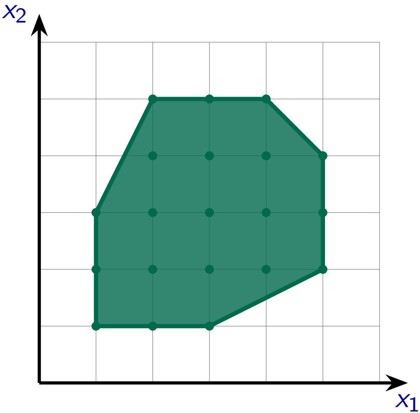

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$\longrightarrow$ LP Relaxation!

LP Relaxation

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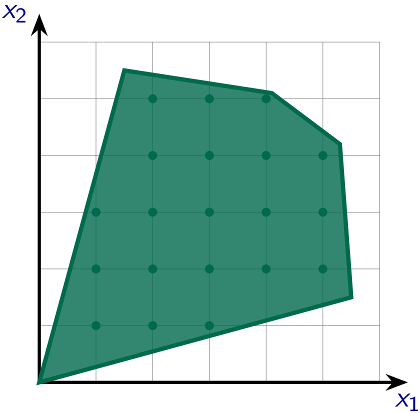
Given an ILP

$$\begin{array}{rrcl} \min & c^T x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \in & \mathbb{N} \end{array}$$

LP relaxation:
Replace $x_v \in \{0, 1\}$ by $x_v \geq 0$.

Observation. $z_{\text{LP}} \leq z_{\text{ILP}}$ (Every ILP solution is feasible for the LP.)



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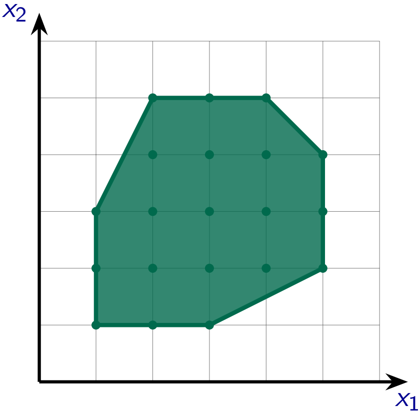

Ideal case: Polyhedron has integral vertices (= integral polyhedron)
$\Rightarrow$ LP relaxation has an integral optimal solution.

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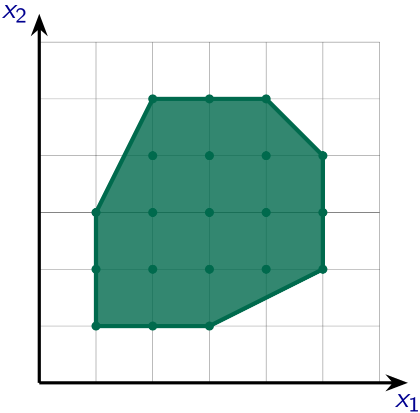

Ideal case: Polyhedron has integral vertices (= integral polyhedron)
⇒ LP relaxation has an integral optimal solution.

Question: When is a polyhedron integral?

ILPs with nice structure:

Totally Unimodular Matrices

When is a polyhedron integral?

Total unimodulare Matrizen

Definition

A matrix $A \in \mathbb{Z}^{m \times n}$ is totally unimodular (TU), if every quadratic submatrix of A has determinants $0, -1$ or $+1$.

A quadratic submatrix $B \in \mathbb{Z}^{k \times k}$ of A is obtained by deleting $m - k$ rows and $n - k$ columns in A.

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$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \text{ is TU}$$

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$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ is not TU, since $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \notin \{-1, 0, 1\}$

Totally Unimodular Matrices

Theorem (Hoffmann, Kruskal 1956)

A matrix $A \in \mathbb{Z}^{m \times n}$ is totally unimodular if and only if the polyhedron $P = \{x \mid Ax \leq b, x \geq 0\}$ has only integral vertices for any $b \in \mathbb{Z}^m$.

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Jackpot!

LP solver can find ILP solution

Characterizations

Lemma (Characterizations)

Let $A \in \{-1, 0, 1\}^{m \times n}$, then the following statements are equivalent:

1. A is totally unimodular.
2. A^T is totally unimodular.
3. There is no quadratic submatrix in A with determinant $+2$ or -2.

[Gomory]

Lemma [Poincaré, 1900]

Let $A \in \{-1, 0, 1\}^{m \times n}$ be a matrix with at most one $+1$ and at most one -1 in each column. Then a A is totally unimodular.

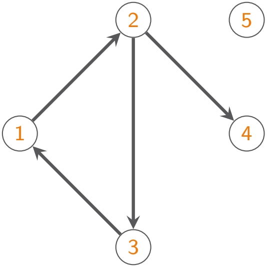
Graph Problems with TU Matrix?

Representation of Graphs

Incidence matrix

The incidence matrix $B \in \mathbb{N}^{m \times n}$ of an (un-)directed graph $G = (V, A)$ is defined as

$$b_{ij} := \begin{cases} 1, & \text{if edge } e_j = (v_i, u) \text{ exists} \\ -1, & \text{if edge } e_j = (u, v_i) \text{ exists (resp. 1 if undirected)} \\ 0, & \text{othw.} \end{cases}$$



$$B = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc} (1,2) & (2,3) & (3,1) & (2,4) \\ \end{array}$$

$$B = \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Totally Unimodular Matrices for Graphs

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Maximum matching in bipartite graphs:
Incidence matrix A is totally unimodular.

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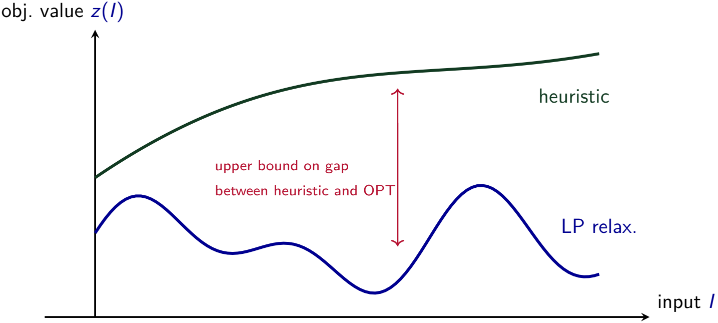
Thus there exists an integral optimal LP solution.

Often LP relaxation is not known to be integral – Still very useful!

The benefit of LP Relaxations

LP relaxation as a lower bound on the optimal solution!

- Useful for evaluating the performance of heuristics



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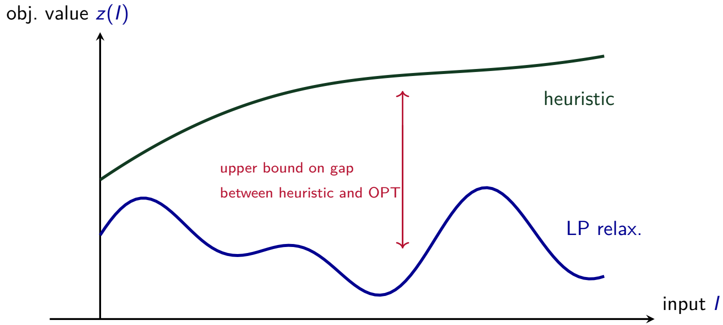

- Use an infeasible LP solution for constructing a (close) feasible integral solution.

Take Home Messages and Outlook

- Linear programs can be solved in polynomial time.

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  - Round LP solution to “good” integral solution (not this course)

Linear Programming is a powerful machinery!