Advanced Algorithms

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Linear Programming II: ILPs and Duality

Lecture 10

LPs are easy, ILPs not

LPs can be solved in polynomial time!

- Interior Point method
- Ellipsoid method

Caution: Not known if Simplex runs in polynomial time
But it works well in practice, implemented in every solver

Optimization = Separation

In general, ILPs cannot be solved in polynomial time, unless P=NP!

Integer Linear Programs

Integer Linear Programs (ILP)

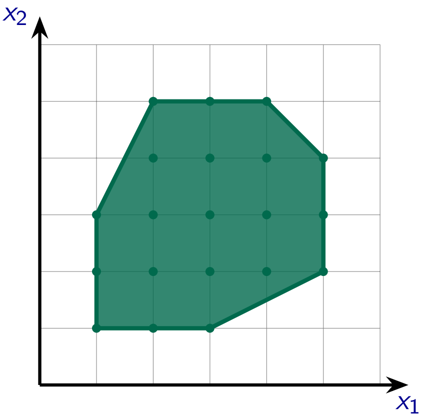
For many problems we find LP formulations only with integrality constraints. $\rightarrow$ Solving ILPs is NP-hard (in general).

Example: Knapsack problem

$$\max \sum_{i=1}^{n} v_i \cdot x_i$$

$$s.t. \sum_{i=1}^{n} w_i \cdot x_i \leq K$$

$$x_i \in \mathbb{Z}_+ \qquad i \in \{1, \ldots, n\}$$



Can we use the machinery of efficiently solving LPs for solving ILPs?

$\longrightarrow$ LP Relaxation!

LP Relaxation

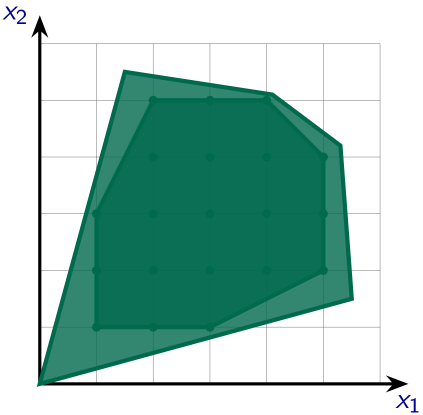
LP Relaxation

Given an ILP

$$\begin{array}{rrcl} \min & c^T x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \in & \mathbb{N} \end{array}$$

LP relaxation:
Replace $x_v \in \{0, 1\}$ by $x_v \geq 0$.

Observation. $z_{\mathrm{LP}} \leq z_{\mathrm{ILP}}$ (Every ILP solution is feasible for the LP.)



Ideal case: Polyhedron has integral vertices (= integral polyhedron)
$\Rightarrow$ LP relaxation has an integral optimal solution.

Question: When is a polyhedron integral?

ILPs with nice structure:

Totally Unimodular Matrices

When is a polyhedron integral?

Total unimodulare Matrizen

Definition

A matrix $A \in \mathbb{Z}^{m \times n}$ is totally unimodular (TU), if every quadratic submatrix of A has determinants $0, -1$ or $+1$.

A quadratic submatrix $B \in \mathbb{Z}^{k \times k}$ of A is obtained by deleting $m-k$ rows and $n-k$ columns in A.

Examples:

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$ is TU

$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ is not TU, since $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \notin \{-1, 0, 1\}$

Totally Unimodular Matrices

Theorem (Hoffmann, Kruskal 1956)

A matrix $A \in \mathbb{Z}^{m \times n}$ is totally unimodular if and only if the polyhedron $P = \{x \mid Ax \leq b, x \geq 0\}$ has only integral vertices for any $b \in \mathbb{Z}^m$.

Corollary

The problem $\min\{c^T x \mid Ax \leq b, x \geq 0\}$ with totally unimodular A and integral b has an integral optimal solution for any c.

Jackpot!

LP solver can find ILP solution

Characterizations

Lemma (Characterizations)

Let $A \in \{-1, 0, 1\}^{m \times n}$, then the following statements are equivalent:

1. A is totally unimodular.
2. A^T is totally unimodular.
3. There is no quadratic submatrix in A with determinant $+2$ or -2.

[Gomory]

Lemma [Poincaré, 1900]

Let $A \in \{-1, 0, 1\}^{m \times n}$ be a matrix with at most one $+1$ and at most one -1 in each column. Then a A is totally unimodular.

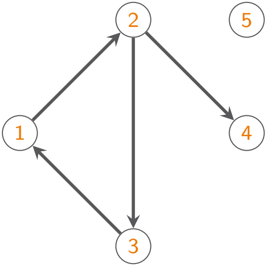
Graph Problems with TU Matrix?

Representation of Graphs

Incidence matrix

The incidence matrix $B \in \mathbb{N}^{m \times n}$ of an (un-)directed graph $G = (V, A)$ is defined as

$$b_{ij} := \begin{cases} 1, & \text{if edge } e_j = (v_i, u) \text{ exists} \\ -1, & \text{if edge } e_j = (u, v_i) \text{ exists (resp. 1 if undirected)} \\ 0, & \text{othw.} \end{cases}$$



$$B = \begin{array}{c} \begin{array}{cccc} (1,2) & (2,3) & (3,1) & (2,4) \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Totally Unimodular Matrices for Graphs

Corollary

The incidence matrix of an undirected graph is totally unimodular if and only if the graph is bipartite.

Korollar

The incidence matrix of a directed graph is totally unimodular.

Maximum matching in bipartite graphs:
Incidence matrix A is totally unimodular.

$$\begin{aligned} \max \quad & 1^T x \\ \text{s.t.} \quad Ax &\leq 1 \\ x &\geq 0 \end{aligned}$$

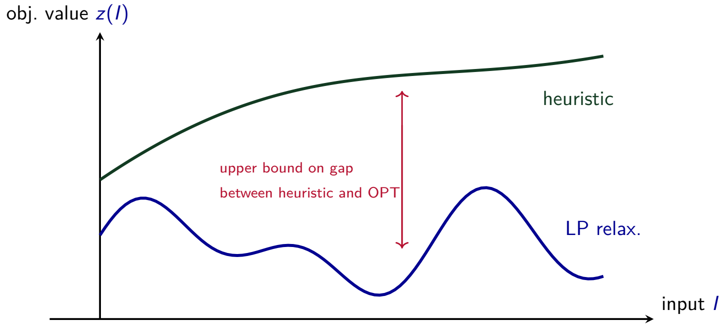
Thus there exists an integral optimal LP solution.

Often LP relaxation is not known to be integral – Still very useful!

The benefit of LP Relaxations

LP relaxation as a lower bound on the optimal solution!

- Useful for evaluating the performance of heuristics



- Use an infeasible LP solution for constructing a (close) feasible integral solution.

LP Duality

Duality: What is it about?

So far: We know linear programs (LPs) and can solve them optimally.

Assume we are given a solution.

- Feasibility is easy to check
- But how good is the solution? How "close" is it to the optimal solution?

The theory of LP duality helps to find upper/lower bounds on the optimal objective value.

Motivation

$$\begin{aligned} \min\ & 7x_1 + 3x_2 && =: z(x) & \\ \text{s.t.}\ & x_1 + x_2 && \geq 2 & (1) \\ & 3x_1 + x_2 && \geq 4 & (2) \\ & x_1, x_2 && \geq 0 & (3) \end{aligned}$$

A feasible solution:
$x_1 = x_2 = 1$ with $z(x) = 10$.
How close to optimum?

Goal: Find lower bound on the optimum.

Ineq. (1),(3) imply: $z(x) = 7x_1 + 3x_2 \geq x_1 + x_2 \geq 2 \Rightarrow \mathrm{OPT} \geq 2$

Ineq. (2),(3) imply: $z(x) = 7x_1 + 3x_2 \geq 3x_1 + x_2 \geq 4 \Rightarrow \mathrm{OPT} \geq 4$

Idea: linear combination of constraints with coefficients $y_1 = 1$ and $y_2 = 2$, that is, $z(x) \geq y_1 \cdot (1) + y_2 \cdot (2)$.

$$z(x) = 7x_1 + 3x_2 \geq 1 \cdot (x_1 + x_2) + 2 \cdot (3x_1 + x_2) \geq 1 \cdot 2 + 2 \cdot 4 = 10.$$

Hence, the above solution is optimal. . . . now generalize.

Motivation

$$\begin{aligned} \min\ & 7x_1 + 3x_2 && =: \ z(x) & \\ \text{s.t.}\ & x_1 + x_2 && \geq 2 & (1) \\ & 3x_1 + x_2 && \geq 4 & (2) \\ & x_1, x_2 && \geq 0 & (3) \end{aligned}$$

Find $y_1 \geq 0$ and $y_2 \geq 0$ with

$$\begin{aligned} z(x) &\geq y_1(x_1 + x_2) + y_2(3x_1 + x_2) \\ &\geq y_1 \cdot 2 + y_2 \cdot 4 \end{aligned}$$

maximizing the right hand side.
It must hold: $y_1 + 3y_2 \leq 7$ and $y_1 + y_2 \leq 3$. It is again an LP.

Primal LP

$$\begin{aligned} \min\ & 7x_1 + 3x_2 & \\ \text{s.t.}\ & x_1 + x_2 && \geq 2 \\ & 3x_1 + x_2 && \geq 4 \\ & x_1, x_2 && \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \max\ & 2y_1 + 4y_2 & \\ \text{s.t.}\ & y_1 + 3y_2 && \leq 7 \\ & y_1 + y_2 && \leq 3 \\ & y_1, y_2 && \geq 0 \end{aligned}$$

Primal and Dual Program

Arbitrary linear program:

$$
\begin{array}{lll}
\min & c^T \cdot x & \\
\text{s.t.} & a_i \cdot x \geq b_i & \text{für } i \in M_1 \\
 & a_i \cdot x \leq b_i & \text{für } i \in M_2 \\
 & a_i \cdot x = b_i & \text{für } i \in M_3 \\
 & x_j \geq 0 & \text{für } j \in N_1 \\
 & x_j \leq 0 & \text{für } j \in N_2 \\
 & x_j \text{ frei} & \text{für } j \in N_3
\end{array}
$$

Obtain lower bound:

$$
\begin{array}{lll}
\max & b^T \cdot y & \\
\text{s.t.} & y_i \geq 0 & \text{für } i \in M_1 \\
 & y_i \leq 0 & \text{für } i \in M_2 \\
 & y_i \text{ frei} & \text{für } i \in M_3 \\
 & A_j^T \cdot y \leq c_j & \text{für } j \in N_1 \\
 & A_j^T \cdot y \geq c_j & \text{für } j \in N_2 \\
 & A_j^T \cdot y = c_j & \text{für } j \in N_3
\end{array}
$$

Note: a_i denotes row i in matrix A und A_j the column j in A.

The linear program on the right is the dual linear program of the primal linear program on the left. $\rightarrow$ Example at the board

Primal & dual Variables & Constraints

	primal LP (min)	dual LP (max)	
	$\geq b_i$	≥ 0	
Constraints	$\leq b_i$	≤ 0	Variables
	$= b_i$	free	
	≥ 0	$\leq c_i$	
Variables	≤ 0	$\geq c_i$	Constraints
	free	$= c_i$	

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Examples

primal LP (min)	dual LP (max)
$\min\ c^T x$ s.t. $Ax \geq b$ $x \geq 0$	$\max\ b^T y$ s.t. $A^T y \leq c$ $y \geq 0$
$\min\ c^T x$ s.t. $Ax = b$ $x \geq 0$	$\max\ b^T y$ s.t. $A^T y \leq c$

Lemma. The dual LP of a dual LP is the primal LP.

Duality Theorems

Weak Duality

primal (P) $\min\ c^T x$ s.t. $Ax \geq b$, $x \geq 0$

dual (D) $\max\ b^T y$ s.t. $A^T y \leq c$, $y \geq 0$

$$\begin{array}{llll} \text{primal (P)} & \min & c^T x \\ & \text{s.t.} & Ax \geq b \\ & & x \geq 0 \end{array} \qquad \begin{array}{llll} \text{dual (D)} & \max & b^T y \\ & \text{s.t.} & A^T y \leq c \\ & & y \geq 0 \end{array}$$

Theorem

Let $\bar{x}$ be feasible a feasible solution for the primal LP (P) and let $\bar{y}$ be a feasible solution for the dual LP (D). Then

$$c^T \cdot \bar{x} \ \geq \ \bar{y}^T \cdot b.$$

Proof. $c^T \cdot \bar{x} \geq (A^T \bar{y})^T \cdot \bar{x} = \bar{y}^T A \cdot \bar{x} \geq \bar{y}^T \cdot b$ $\square$

1. If (P) is unbounded (Opt $= -\infty$), then (D) is infeasible.
2. If (D) is unbounded (Opt $= \infty$), then (P) is infeasible.
3. Let $\bar{x}$ and $\bar{y}$ be feasible solutions for (P) and (D) with $c^T \cdot \bar{x} = \bar{y}^T \cdot b$, then $\bar{x}$ and $\bar{y}$ are optimal.

Strong Duality

Theorem

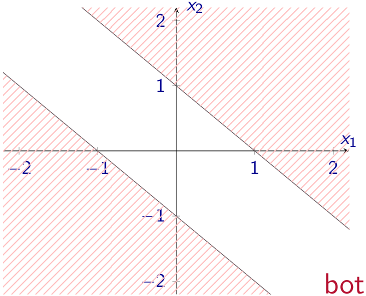
If the primal LP has an optimal solution x^*, then there exists an optimal solution y^* for the dual LP and $c^T x^* = b^T y^*$.

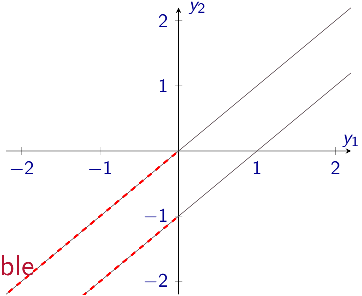
Possible primal-dual pairs:

primal\dual	optimal	unbounded	infeasible
optimal	strong duality	impossible	impossible
unbounded	impossible	impossible	(1) weak duality
infeasible	impossible	(1) weak duality	(2) possible, c.f. Ex.

Example I

$$
\begin{array}{llcrcrl}
\max & x_1 & & & & & \\
\text{s.t.} & x_1 & + & x_2 & \geq & 1 & \mid y_1 \\
 & -x_1 & - & x_2 & \geq & 1 & \mid y_2 \\
 & & & x_1,\, x_2 & \in & \mathbb{R} &
\end{array}
\qquad
\begin{array}{llcrcrl}
\min & y_1 & + & y_2 & & & \\
\text{s.t.} & y_1 & - & y_2 & = & 1 & \mid x_1 \\
 & y_1 & - & y_2 & = & 0 & \mid x_2 \\
 & & & y_1,\, y_2 & \leq & 0 &
\end{array}
$$

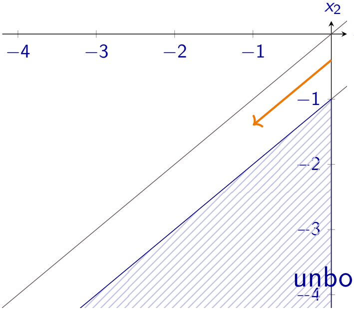


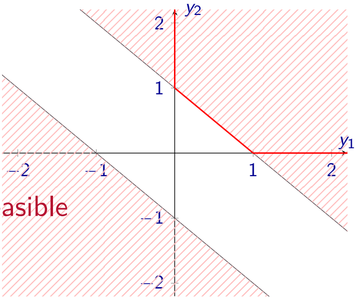


!
both infeasible

Example II

$$
\begin{array}{llcllll}
\min & x_1 & + & x_2 & & & \\
\text{s. t.} & x_1 & - & x_2 & \geq & 1 & \mid y_1 \\
 & x_1 & - & x_2 & \geq & 0 & \mid y_2 \\
 & & & x_1, x_2 & \leq & 0 &
\end{array}
\qquad
\begin{array}{llcllll}
\max & y_1 & & & & & \\
\text{s. t.} & y_1 & + & y_2 & \geq & 1 & \mid x_1 \\
 & -y_1 & - & y_2 & \geq & 1 & \mid x_2 \\
 & & & y_1, y_2 & \geq & 0 &
\end{array}
$$





⚠ infeasible

unbounded ⚠

An application of duality

Minimal Vertex Cover

Problem: Min Vertex Cover

Given: Graph $G = (V, E)$
Task: Find a minimal subset $V' \subseteq V$ such that each edge $e = \{u, v\} \in E$ has an endpoint in V'; we say e is covered.

Integer LP: Decision variable $x_v \in \{0, 1\}$ indicates if $v \in V'$.

$$\begin{aligned} \min \quad & \sum_{v \in V} x_v =: z \\ \text{s.t.} \quad & x_u + x_v \geq 1, && \text{for all } \{u, v\} \in E \\ & x_v \in \{0, 1\}, && \text{for all } v \in V. \end{aligned}$$

LP relaxation: Replace $x_v \in \{0, 1\}$ by $x_v \geq 0$.

Observation: $z_{\text{LP}} \leq z_{\text{ILP}}$ (Any ILP solution is feasible for the LP.)

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Dual LP for Minimal Vertex Cover

The dual LP for the LP relaxation (board):

$$\begin{aligned} \max \quad & \sum_{e \in E} y_e & & \\ \text{s.t.} \quad & \sum_{e \in \delta(v)} y_e \leq 1, & & \text{for all } v \in V \\ & y_e \geq 0, & & \text{for all } e \in E. \end{aligned}$$

For $v \in V$ let $\delta(v) := \{e \in E \mid e = (u, v), u \in V\}$ the set of edges that are incident with v.

Observation. LP Relaxation of the Maximal Matching.

Problem: Max Matching

Given: Graph $G = (V, E)$

Task: Find a maximal matching, i.e., a maximal subset $M \subseteq E$ such that any vertex is incident to at most one edge $e \in M$.

König's Theorem

Theorem

There are at least as many vertices in a minimal vertex cover as there are edges in a maximal matching.

Proof. Follows by weak duality: $z_{\mathrm{VC}} \geq z_{\mathrm{VC}}^{\mathrm{LP}} = z_{\mathrm{M}}^{\mathrm{LP}} \geq z_{\mathrm{M}}$.

Definition. A Graph $G = (V, E)$ is bipartite, if there exists a partition $V = L \cup R$ such that there are no edges between L and R, i.e., there exists no $\{u, v\} \in E$ with $u \in L$ and $v \in R$.

Theorem (König, 1931)

In a bipartite graph it holds that the number of vertices in a minimal vertex cover **equals** the number of edges in a maximal matching.

Important: In general, weak and strong duality only hold for LPs! However, it can be shown that the LP relaxations of the vertex cover and matching ILPs always have an integral solution if the graph is bipartite. (constraint matrix is totally unimodular)

Complementary slackness

Complementary Slackness

Consider an arbitrary primal-dual pair (P) and (D):

$$(P)\ \min\ c^T \cdot x \quad \text{s.t. } A \cdot x \geq b,\ x \geq 0$$

$$(D)\ \max\ y^T \cdot b \quad \text{s.t. } A^T \cdot y \leq c,\ y \geq 0$$

Theorem

Let $\bar{x}$ be feasible for (P) and $\bar{y}$ feasible for (D). Then, $\bar{x}$ and $\bar{y}$ are optimal if and only if

$$\bar{x}_i \cdot (c_i - (A^T \cdot \bar{y})_i) = 0, \qquad \text{for all } i, \text{ and}$$

$$\bar{y}_j \cdot (b_j - (A \cdot \bar{x})_j) = 0, \qquad \text{for all } j.$$

The theorem holds for arbitrary prima-dual pairs $(P), (D)$.

Corollary. In optimal solution: either, a variable vanishes ($= 0$), or, the corresponding dual inequality is tight ($=$). A free variable corresponds to an equation in the dual that is tight by definition.

Complementary Slackness for Proving Optimality

Given feasible solutions $\bar{x}$ und $\bar{y}$ for a primal LP (P) and its dual LP (D), we can use complementary slackness to verify optimality.

Example at the board.

Faster: Compare objective function values $\bar{x}$ und $\bar{y}$ and use duality.

Complementary Slackness for Proving Optimality

Given an optimal solution x^* for a primal LP (P), use complementary slackness to construct an optimal solution y^* for the dual LP:

1. If constraint i in (P) is not tight, then set $y_i^* = 0$.
2. For all x_j^* with $x_j^* \neq 0$, the correspond. constraint must be tight:
   Set up a system of linear equations and solve it to determine the remaining duals
3. These duals give an optimal solution to (D).

Example below with solution at the board:

$$
\begin{array}{rrl}
(P) & \min & 5x_1 + 12x_2 + 2x_3 \\
& \text{s.t.} & 2x_1 + 4x_2 - 8x_3 \leq 1 \\
& & -8x_1 + 4x_2 - x_3 \geq 0 \\
& & x_1 + 2x_2 + 2x_3 = 5 \\
& & x_1, x_3 \geq 0
\end{array}
$$

$$
\begin{array}{rrl}
(D) & \max & y_1 + 5y_3 \\
& \text{s.t.} & 2y_1 - 8y_2 + y_3 \leq 5 \\
& & 4y_1 + 4y_2 + 2y_3 = 12 \\
& & -8y_1 - 4y_2 + 2y_3 \leq 2 \\
& & y_1 \leq 0 \\
& & y_3 \geq 0
\end{array}
$$

The optimal solution is $x^* = (0, \frac{1}{2}, 2)^T$.

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Applications of Duality

Dual Variable as Shadowprices

Shadow prices

Consider a maximization problem. The values of the dual variables y_i can be interpreted as the value of one unit of resource i, as limited by the i-th constraint.

Example in Exercise.

Primal-Dual Algorithms (General)

Idea. Solve the LP by simultaneously generating a dual feasible solution. Use complementary slackness to improve the solution.

1. Set primal LP variables $x = 0$ and dual variables $y = 0$. (dual feasible, primal not necessarily)
2. Increase some of the y_j's until a dual constraint is tight. Freeze the corresponding y_j and increase the corresponding x.
3. Repeat step 2 until all constraints are satisfied.

Weighted Vertex Cover - LP Relaxation

Every vertex has a weight $w_v \geq 0$, for all $v \in V$.

Goal: Find a vertex cover V' with minimal total weight.

$$\begin{aligned} \min \quad & \sum_{v \in V} w_v \cdot x_v \\ \text{s.t.} \quad & x_u + x_v \geq 1, \ \forall (u,v) \in E \\ & x_v \geq 0, \ \forall v \in V. \end{aligned}$$

$$\begin{aligned} \max \quad & \sum_{e \in E} y_e \\ \text{s.t.} \quad & \sum_{e \in \delta(v)} y_e \leq w_v, \ \forall v \in V \\ & y_e \geq 0, \quad \forall e \in E. \end{aligned}$$

Primal-Dual Algorithm

1. Set primal LP variables $x = 0$ and dual variables $y = 0$.
2. Choose any unfrozen edge e.
3. Increase y_e until a constraint is satisfied. $\rightarrow$ for vertex v
4. Freeze all y_e with $e \in \delta(v)$. Increase $x_v = 1$.
5. Repeat from step 2 until all y_e are frozen.

Primal-Dual Algorithm for Vertex Cover

Theorem

The primal-dual algorithm finds a feasible solution x for the minimum vertex cover problem with

$$z(x) \leq 2 \cdot z(x^*_{\mathrm{LP}}) \leq 2 \cdot z(x^*_{\mathrm{ILP}}),$$

where x^* is the optimal solution. This is called a 2-approximation.

Proof. . . . on the board

Remark: No polynomial-time algorithm is known that achieves a $(2-\varepsilon)$-approximation for any $\varepsilon > 0$. Assuming the Unique Games Conjecture (complexity theory), no such algorithm exists.

Alternative 2-approximation: rounding the LP solution.

Take-Home Message

- Duality is a powerful concept for obtaining
  - bounds on the value of an optimal solution OPT
  - an estimate on the gap between a feasible solution and OPT
- Duality theorems
- Applications of duality
  - Duality of problems: min vertex cover and max matching; max flow and min cut; etc.
  - Complementary slackness
  - Shadow prices, interpretation of duals for detecting bottlenecks
- Duality theory only for linear programs!

This was a crash course on LP basics. More details and applications in courses: Operations Research, Optimization Bootcamp, Master courses.