Advanced Algorithms

Nicole Megow (Universität Bremen)

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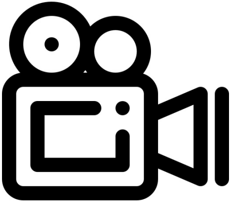
Introduction to Matroids

Lecture 12

Recording of this Lecture

This lecture will be recorded

- Recording only of the lecturers by themselves.
- If there are questions from the audience, please make a clear signal if the microphone shall be muted.
- Our goal is to record the lecture, but it is no guarantee that each lecture will be recorded.



Recap: Minimum Spanning Trees

Given: undirected graph $G = (V, E)$ with edge weights $w : E \to \mathbb{R}$.

Task: Find a spanning tree $T \subseteq E$ of G with minimum weight, that is, $\sum_{e \in T} w(e)$ is minimal.

Kruskal's algorithm

$F \leftarrow \emptyset$
for $e \in E$ sorted ascending by $w(e)$ **do**
  if $F \cup \{e\}$ is acyclic **then**
    $F \leftarrow F \cup \{e\}$
return F

Can we apply the idea of Kruskal's algorithm to other problems?

Can we generalize Kruskal's algorithm?

A Generalization of Kruskal's Algorithm

Given: universe E with weights $w : E \to \mathbb{R}$, feasible sets $\mathcal{I} \subseteq 2^E$.

Task: Find a $F \in \mathcal{I}$, that is, $\sum_{e \in F} w(e)$ is minimal / maximal.

An abstraction of Kruskal's algorithm

$F \leftarrow \emptyset$
for $e \in E$ sorted ascending by $w(e)$ **do**
  if $F \cup \{e\} \in \mathcal{I}$ (that is, $F \cup \{e\}$ is feasible) **then**
    $F \leftarrow F \cup \{e\}$
return F

Observation: if $I \in \mathcal{I}$ is a feasible set, then every $I' \subseteq I$ should also be feasible.

Independence Systems

Definition (Independence System)

Let E be a ground set. A set system $\mathcal{I} \subseteq 2^E$ is called independence system if

(i) $\emptyset \in \mathcal{I}$, and

(ii) for all $A \in \mathcal{I}$ and $B \subseteq A$, we have $B \in \mathcal{I}$.

A set $A \subseteq E$ is called independent if $A \in \mathcal{I}$, and dependent if $A \notin \mathcal{I}$. Minimal dependent sets are called circuits, and maximal independent sets are called bases.

For some set $A \subseteq E$, we call a maximum independent subset of A a basis of A.

Examples of Independence Systems

- **Minimum spanning tree:** $w(e)$ is the weight of edge e, and $\mathcal{I} = \{I \subseteq E \mid I \text{ forest}\}$.
- **Maximum matching:** $w(e) = 1$, and $\mathcal{I} = \{I \subseteq E \mid I \text{ matching}\}$.
- **Knapsack:** $w(e)$ is the value of item e, and $\mathcal{I} = \{I \subseteq E \mid \sum_{e \in I} w(e) \leq B\}$ for some capacity B.
- **Maximum weight independent set:** $w(v)$ is the weight of vertex v, and $\mathcal{I} = \{I \subseteq V \mid I \text{ is independent in } G\}$.

Matroids

Definition (Matroid)

An independence system $(E, \mathcal{I})$ is called a matroid if

(iii) for all $A, B \in \mathcal{I}$ with $|A| < |B|$, there exists an element $b \in B \setminus A$ such that $A \cup \{b\} \in \mathcal{I}$. (augmentation property)

In summary, we have the properties:

(i) $\emptyset \in \mathcal{I}$, and

(ii) for all $A \in \mathcal{I}$ and $B \subseteq A$, we have $B \in \mathcal{I}$.

(iii) for all $A, B \in \mathcal{I}$ with $|A| < |B|$, there exists an element $b \in B \setminus A$ such that $A \cup \{b\} \in \mathcal{I}$. (augmentation property)

Bases of a Matroid

Lemma

Every base of a matroid has the same size.

Proof.

- B_1 and B_2 bases such that $|B_1| < |B_2|$.
- By the augmentation property, $\exists b \in B_2 \setminus B_1$ such that $B_1 \cup \{b\} \in \mathcal{I}$.
- Contradiction to the maximality of B_1.

□

Uniform Matroids

Let E be a universe and let $k \in \mathbb{N}$. Define

$$\mathcal{I} = \{I \subseteq E \mid |I| \leq k\}.$$

Theorem

$\mathcal{M} = (E, \mathcal{I})$ is a matroid (called a uniform matroid).

Partition Matroids

Let E be a ground set and let $E_1, E_2, \ldots, E_\ell$ be a partition of E. For fixed integers $k_1, k_2, \ldots, k_\ell$, define

$$\mathcal{I} = \{I \subseteq E \mid |I \cap E_i| \leq k_i \text{ for all } 1 \leq i \leq \ell\}.$$

Theorem

$\mathcal{M} = (E, \mathcal{I})$ is a matroid (called a partition matroid).

Linear Matroids

Let F be a field and let $A \in F^{m \times n}$ be a matrix whose columns are indexed by a ground set $E = \{1, \ldots, n\}$. Define

$$\mathcal{I} = \{I \subseteq E \mid A_I \text{ has full rank}\},$$

where A_I denotes the submatrix of A of the columns indexed by I.

Theorem

$\mathcal{M} = (E, \mathcal{I})$ is a matroid (called a linear matroid).

Graphic Matroids

Let $G = (V, E)$ be a graph. Define

$$\mathcal{I} = \{I \subseteq E \mid I \text{ is acyclic}\} .$$

Theorem

$\mathcal{M} = (E, \mathcal{I})$ is a matroid (called a graphic matroid).

In particular, this shows that the problem of finding a MST in a graph is a matroid optimization problem.

Matching Matroids

Let $G = (V, E)$ be a graph. Define

$$\mathcal{I} = \{I \subseteq V \mid \text{there exists a matching in } G \text{ that covers } I\}.$$

Theorem

$\mathcal{M} = (V, \mathcal{I})$ is a matroid (called a matching matroid).

The Rank of a Matroid

Definition

Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. The rank function $r : 2^E \to \mathbb{N}_{\geq 0}$ associated to $\mathcal{M}$ is defined as

$$r(S) := \max_{I \subseteq S \text{s.t.} I \in \mathcal{I}} |I|$$

for each $S \subseteq E$. We call $r(E)$ the rank of $\mathcal{M}$.