Exercises for the lecture

Mathematics III

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Solutions to Exercise 02

Tasks

5. **Elementary probability theory.**

(a)

$$\begin{aligned}\text{The required probability } &= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &+ P(A \cap B \cap C).\end{aligned}$$

where

$$P(A \cap B \cap C) = 0.1, \quad P(C) = 0.4, \quad P(B \cap A^c) = 0.3$$
$$P(C \cap A^c) = 0.15, \quad P(A^c \cap B \cap C^c) = 0.2$$
$$P(A \cap B^c) = 0.25, \quad P(A \cap C^c) = 0.25$$

$$\begin{aligned}P(C) &= P(A \cap C) + P(A^c \cap C) \\ 0.4 &= P(A \cap C) + 0.15 \\ P(A \cap C) &= 0.25.\end{aligned}$$

$$\begin{aligned}P(A) &= P(A \cap C^c) + P(A \cap C) \\ &= 0.25 + 0.25 \\ &= 0.5.\end{aligned}$$

$$\begin{aligned}P(B \cap C) &= P(B \cap A^c) + P(A \cap B \cap C) - P(A^c \cap B \cap C^c) \\ P(B \cap C) &= 0.3 + 0.1 - 0.2 \\ P(B \cap C) &= 0.2\end{aligned}$$

$$\begin{aligned}P(A \cap B) &= P(A) - P(B^c \cap A) \\ P(A \cap B) &= 0.5 - 0.25 \\ P(A \cap B) &= 0.25\end{aligned}$$

$$\begin{aligned}P(B) &= P(B \cap A) + P(B \cap A^c) \\ P(B) &= 0.25 + 0.3 \\ P(B) &= 0.55.\end{aligned}$$

so the probability is $0.5 + 0.55 + 0.4 - 0.25 - 0.25 - 0.2 + 0.1 = 0.85$.

(b) The required probability is

$$P(A\cap B)+P(A\cap C)+P(B\cap C)-3P(A\cap B\cap C).$$

$$=0.25+0.25+0.2-3(0.1)$$

$$=0.4.$$

(c) The required probability is 1−(Prob. that at least one event occurs) = 1−(Prob. in part (a) above).

$$=1-0.85,$$

$$=0.15.$$

(d) The required probability is

Prob(0 events occur)+Prob(exactly one event occurs),

where

Prob(0 events occur) = The prob. in part (c) above,

and

Prob(exactly one event occurs) = Prob. in part (a) above-Prob. in part (b) above-$P(A\cap B\cap C)$.

$$=0.85-0.4-0.1,$$

$$=0.35.$$

6. **Combinatorics**

Solutions when the order of groups is important:

(a) Number of ways of choosing the first group = $\binom{9}{3}$.
Number of ways of choosing the second group = $\binom{6}{3}$.
Number of ways of choosing the third group = $\binom{3}{3}$.
Total number of ways = $\binom{9}{3}\binom{6}{3}\binom{3}{3}=1,680$.

Alternatively: The number of ways in which mn different items can be divided equally into m groups, each containing n objects

$$=\left[\frac{(mn)!}{(n!)^m}\right],$$

where

$$m=3 \text{ and } n=3,$$

$$=\left[\frac{(3*3)!}{(3!)^3}\right],$$

$$=1,680.$$

(b) The number of ways in which $(m+n+p)$ different students can be divided into three different groups containing m, n, and p students, respectively

$$= \frac{(m+n+p)!}{m!n!p!} * 3!.$$

where

$$(m,n,p) = \{(2,3,4),(2,4,3),(3,2,4),(3,3,3),(3,4,2),(4,2,3),(4,3,2)\},$$

i.e., sum the different combinations above.

(c) The number of ways in which $(m+n+p)$ different students can be divided into three different groups containing m, n, and p students, respectively

$$= \frac{(m+n+p)!}{m!n!p!} * 3!,$$

where

$$(m,n,p) = \{(1,1,7),(1,2,6),(1,3,5),(1,4,4),(1,5,3),(1,6,2),(1,7,1),(2,1,6),(2,2,5),(2,3,4),$$
$$(2,4,3),(2,5,2),(2,6,1),(3,1,5),(3,2,4),(3,3,3),(3,4,2),(3,5,1),(4,1,4),(4,2,3),(4,3,2),$$
$$(4,4,1),(5,1,3),(5,2,2),(5,3,1),(6,1,2),(6,2,1),(7,1,1)\}.$$

i.e., sum the different combinations above.

Solutions when order of the groups is not important:

(a) The number of ways in which mn different items can be divided equally into m groups, each containing n objects

$$= \left[\frac{(mn)!}{(n!)^m}\right]\frac{1}{m!},$$

where

$$m = 3 \text{ and } n = 3,$$

$$= \left[\frac{(3*3)!}{(3!)^3}\right]\frac{1}{3!},$$

$$= 280.$$

(b) The number of ways in which $(m+n+p)$ different students can be divided into three different groups containing m, n, and p students, respectively

$$= \frac{(m+n+p)!}{m!n!p!}.$$

where

$$(m,n,p) = \{(2,3,4),(3,3,3)\},$$

i.e., sum the different combinations above.

(c) The number of ways in which $(m+n+p)$ different students can be divided into three different groups containing m, n, and p students, respectively

$$= \frac{(m+n+p)!}{m!n!p!},$$

where

$$(m,n,p) = \{(1,1,7),(1,2,6),(1,3,5),(1,4,4),(2,2,5),(2,3,4),(3,3,3)\}.$$

i.e., sum the different combinations above.

7. **Conditional Probabilities**
   Let

   - $P(D)$ denote the prob. of having a disease,
   - $P(D^c)$ denote the prob. of not having a disease,
   - $P(+)$ denote the prob. of the test giving a positive result, and
   - $P(-)$ denote the prob. of the test giving a negative result. Then,

$$P(D^c) = 0.999, \quad P(D) = 0.001,$$

$$P(+|D) = 0.999, \quad P(+|D^c) = 0.002,$$

$$P(D|+) = \frac{P(+|D)P(D)}{P(+|D)P(D) + P(+|D^c)P(D^c)},$$

$$= \frac{(0.999 * 0.001)}{(0.999 * 0.001) + (0.002 * 0.999)}.$$

8. **Multiple Select Task**

   (a) TRUE. Since $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{25}{24} > 1$ which can never be the case.

   (b)

   (c)

   (d) TRUE. Since $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and $P(A \cap B) \neq 0$.