University of Bremen

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Exercises for the lecture

Mathematics III

Summer Semester 2023

Solutions to Exercise 04

Tasks

13. **Binomial distribution**
Let X be the number of passengers who show up for the flight, then $X \sim Bin(n, p)$ where $n = 270$ and $p = 0.96$.

We need,

$$P(X > 264) = P(X = 265)+, \ldots, +P(X = 270),$$

$$= \binom{270}{265}(0.96)^{265}(1-0.96)^{270-265} + \ldots + \binom{270}{270}(0.96)^{270}(1-0.96)^{270-270}.$$

14. **Poisson distribution**
Let $V_n \sim B(n, p)$ be the number of manuscripts accepted for publication (and N be the number submitted for publication), we need to find the distribution of $Y = V_N$ where $N \sim Pois(\lambda)$ random variable and is independent of $V_1, V_2, \ldots$. For $n = 1, 2, \ldots$, using the properties of Binomial distribution, then

$$V_n = X_1 + X_2 + \ldots + X_n,$$

where $X_1, X_2, \ldots$ are independent Bernoulli random variables so that Y is a sum of Bernoulli random variables ($X = 1$ if a manuscript is accepted for publication and $X = 0$ otherwise.)

$$Y = X_1 + X_2 + \ldots + X_n,$$

Using

$$P(s) = E(S^{X_k}) = q + ps, \; k = 1, 2, \ldots,$$

and

$$Q(s) = E(S^N) = e^{\lambda(s-1)},$$

therefore

$$R(s) = E(S^Y) = Q(P(s)) = exp\{\lambda(ps - p)\} = exp\{\lambda p(s - 1)\}$$

which is the PGF for a Poisson random variable with parameter λp.

<u>Alternative Solution:</u>

Musterlösung zu Aufgabe 14:

Seien X = Anzahl eingereichte Manuskripte,
Y = Anzahl veröffentlichter Manuskripte.
(beides $\mathbb{N}_0$-wertige Zufallsvariablen)

Gegeben: (i) $\mathcal{L}(X) = \text{Poisson}(\lambda)$,

d.h.

$$\mathbb{P}(X=n) = \frac{\lambda^n}{n!} \exp(-\lambda), \quad n \in \mathbb{N}_0$$

(ii) $\mathcal{L}(Y \mid X=n) = \text{Bin}(n,p)$,

d.h.

$$\mathbb{P}(Y=k \mid X=n) = \binom{n}{k} p^k (1-p)^{n-k}, \quad 0 \leq k \leq n$$

Zu zeigen: $\mathcal{L}(Y) = \text{Poisson}(\lambda \cdot p)$, d.h.

$$\forall m \in \mathbb{N}_0: \mathbb{P}(Y=m) = \frac{(\lambda p)^m}{m!} \exp(-\lambda m).$$

Sei $m \in \mathbb{N}_0$ beliebig, aber fest.
Wir rechnen:

$$\mathbb{P}(Y=m) = \sum_{n=m}^{\infty} \mathbb{P}(Y=m, X=n)$$

↑

Zerlegungsformel } Y kann niemals größer als X sein!

$$= \sum_{n=m}^{\infty} \mathbb{P}(Y=m \mid X=n) \circ \mathbb{P}(X=n)$$

$$= \sum_{n=m}^{\infty} \binom{n}{m} p^m (1-p)^{n-m} \cdot \frac{\lambda^n}{n!} \exp(-\lambda)$$

$$= \left(\frac{p}{1-p}\right)^m \exp(-\lambda) \sum_{n=m}^{\infty} \frac{1}{m!\,(n-m)!} [(1-p)\lambda]^n$$

$$= \left(\frac{p}{1-p}\right)^m \frac{1}{m!} \exp(-\lambda) \sum_{n=m}^{\infty} \frac{[(1-p)\lambda]^n}{(n-m)!}$$

$$= \left(\frac{p}{1-p}\right)^m \cdot \frac{1}{m!} \exp(-\lambda) \sum_{n=m}^{\infty} \frac{[(1-p)\lambda]^m [(1-p)\lambda]^{n-m}}{(n-m)!}$$

$$= \frac{(p\lambda)^m}{m!} \exp(-\lambda) \exp((1-p)\lambda) = \frac{(p\lambda)^m}{m!} \exp(-\lambda p).$$

☒

15. **Distribution function**

(a)

$$\begin{aligned} F(-2) &= f(-2), \\ \therefore f(-2) &= 1/4. \end{aligned}$$

$$\begin{aligned} F(-1/2) &= f(-2) + f(-1/2), \\ 1/2 &= 1/4 + f(-1/2), \\ \therefore f(-1/2) &= 1/4. \end{aligned}$$

$$\begin{aligned} F(-1/2) &= f(-1/2) + f(1/4) + f(-2), \\ 1/2 &= 1/4 + 1/4 + f(-1/2), \\ \therefore f(-1/2) &= 0. \end{aligned}$$

$$\begin{aligned} F(3/7) &= f(3/7) + f(-1/2) + f(1/4) + f(-2), \\ 4/5 &= 0 + 1/4 + 1/4 + f(3/7), \\ \therefore f(3/7) &= 3/10. \end{aligned}$$

$$\begin{aligned} F(8/11) &= f(8/11) + f(3/7) + f(-1/2) + f(1/4) + f(-2), \\ 1 &= 3/10 + 0 + 1/4 + 1/4 + f(8/11), \\ \therefore f(8/11) &= 1/5. \end{aligned}$$

X	-2	-1/2	3/7	8/11
$P(X)$	1/4	1/4	3/10	1/5

(b) i. Using CDF,

$$\begin{aligned} P(-1 < X \leq 1) &= F(1) - F(-1), \\ &= 1 - 1/4, \\ &= 3/4. \end{aligned}$$

$$\begin{aligned} P(-1/2 \leq X < 8/11) &= F(3/7) - F(-1/2), \\ &= 4/5 - 1/2, \\ &= 3/10. \end{aligned}$$

$$\begin{aligned} P(-2 < X < 3/7) &= F(-1/2) - F(-2), \\ &= 1/2 - 1/4, \\ &= 1/4. \end{aligned}$$

ii. Using PDF,

$$\begin{aligned} P(-1 < X \leq 1) &= f(-1/2) + f(3/7) + f(8/11), \\ &= 3/10 + 1/5 + 1/4, \\ &= 3/4. \end{aligned}$$

$$\begin{aligned} P(-1/2 \leq X < 8/11) &= f(-1/2) + f(3/7), \\ &= 3/10 + 1/4, \\ &= 3/10. \end{aligned}$$

$$\begin{aligned} P(-2 < X < 3/7) &= f(-1/2), \\ &= 1/4. \end{aligned}$$

16. **Multiple select task**

(a) TRUE. As the table below illustrates.

	Without replacement	With replacement
Ordered	$\frac{n!}{(n-k)!}$	n^k
Unordered	$\binom{n}{k}$	$\binom{n+k-1}{k}$

Tabelle 1: Number of possible arrangements of size k from n objects

(b) FALSE. Since $nC_k = \binom{n}{k}$ and $nP_k = \frac{n!}{(n-k)!}$.

(c) TRUE. Since $nC_k = \binom{n+k-1}{k}$ and $nP_k = n^k$.

(d) FALSE. Since $nP_k = n^k$.