Exercises for the lecture

Mathematics III

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University of Bremen

Institute for Statistics

Daniel Ochieng
MZH, Room 7240
E-Mail: dochieng@uni-bremen.de

Solutions to Exercise 05

Tasks

17. **Geometric distribution**

Let X be the number of failures ("success in our case"), then

$$f(x) = \theta(1-\theta)^x,$$

To find the MLE,

$$\begin{aligned}
\mathcal{L}(\theta; x) &= \prod_{x=0}^{n} f(x), \\
&= \theta^n (1-\theta)^{\sum_{x=0}^{n} x_i}, \\
\mathcal{L}(\theta; x) &= n\ ln\theta + \sum_{x=0}^{n} x_i\ ln(1-\theta), \\
\frac{\partial}{d\theta}\mathcal{L}(\theta; x) &= \frac{n}{\theta} - \frac{\sum_{x=0}^{n} x_i}{(1-\theta)}, \\
\therefore \widehat{\theta} &= \frac{n}{n + \sum_{x=0}^{n} x_i}, \\
&= \frac{51}{51+50}.
\end{aligned}$$

18. **Geometric probability**
Let t_1 and t_2 be the arrival times for the first and second student, respectively. We need to find $P(A)$ where $A = \{(t_1, t_2) \in \Omega : |t_1 - t_2| \leq 15\}$ and the total sample space Ω is given by the Cartesian product $[0, 30] \times [0, 30]$.

Then,

$$P(A) = \frac{\text{Area (A)}}{\text{Total area}} = \frac{900-225}{900} = \frac{3}{4}.$$

Alternatively:

Prob. that person A arrives at 12:15 is 15/30. Prob. that person B arrives in the next 15 minutes is 15/30. Denote these two events by AB. Hence, $P(AB) = 15/30 * 15/30 = 1/4$.

Prob. that person B arrives at 12:15 is 15/30. Prob. that person A arrives in the next 15 minutes is 15/30. Denote these two events by BA. Hence, $P(BA) = 15/30 * 15/30 = 1/4$.

Prob. that they both arrive after 12:15 is $15/30 * 15/30 = 1/4$.

The total required probability $= 1/4 + 1/4 + 1/4 = 3/4$.

19. **Lebesgue density**

(a) TRUE. We need to prove if $\int_\Omega f(x)dx = 1$. We have,

$$\int_c^\infty \frac{kc^k}{x^{k+1}}dx = \left.\frac{kc^k x^{-k}}{-k}\right|_0^\infty = 1.$$

(b) TRUE. We need to prove if $\int_\Omega f(x)dx = 1$. We have,

$$\int_0^\infty \frac{\alpha^n x^{n-1} e^{-\alpha x}}{(n-1)!}dx = \frac{\alpha^n}{(n-1)!\alpha\alpha^{n-1}} \int_0^\infty u^{n-1}e^{-u}du$$

using $u = \alpha x$ in the change of variable. Using $\Gamma(n) = \int_0^\infty u^{n-1}e^{-u}du$, the above integral reduces to,

$$\frac{\Gamma(n)}{(n-1)!} = 1.$$

20. **Multiple Select Task**

(a) TRUE. Since the random variable is continuous.

(b) TRUE. By the properties of a CDF. If $a < b$ then $F(a) \leq F(b)$.

(c) FALSE. Since $\frac{d}{dx}F_X(x) = f(x) > 0 \; \forall x \in \Omega$ which is not always the case.

(d) FALSE. Since $F_X(-\infty) = 0$ and $F_X(\infty) = 1$ but $-\infty, \infty \notin \mathbb{R}$.