Exercises for the lecture

Mathematics III

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Solutions to Exercise 06

Tasks

21. **Gaussian distribution**

(a) We have $\mu = 3$ and $\delta^2 = 4$ so that,

$$\begin{aligned} P(-2 \leq X \leq 3) &= P\left(\frac{-2-3}{2} \leq Z \leq \frac{3-3}{2}\right), \\ &= P(-2.5 \leq Z \leq 0), \\ &= \phi(0) - \phi(-2.5), \\ &= 0.5000 - 0.00621, \\ &= 0.49379. \end{aligned}$$

(b)

$$\begin{aligned} P(X - 2 < a) &= 0.95, \\ P\left(Z < \frac{a-1}{2}\right) &= 0.95, \\ \phi\left(\frac{a-1}{2}\right) &= 0.95, \\ \frac{a-1}{2} &= \phi^{-1}(0.95), \\ &= 1.64, \\ a &= 4.28. \end{aligned}$$

(c) Using change of variable technique,

$$g(y) = f\left(u^{-1}(y)\right)|J|$$

where

$$\begin{aligned} J &= \frac{d}{dy}\left(u^{-1}(y)\right), \\ f(x) &= \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(\frac{x-\mu}{\sigma}\right)^2, \\ u^{-1}(y) &= y + 2, \\ J &= \frac{d}{dy}(y+2), \\ &= 1, \end{aligned}$$

$$g(y) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\bigg(\frac{y+2-\mu}{\sigma}\bigg)^2, \text{where } \mu = 3 \text{ and } \sigma^2 = 4.$$

22. **Exponential distribution**

$$P(X \leq x_0 + y | X \geq x_0) = \frac{P(X \leq x_0 + y, X \geq x_0)}{P(X \geq x_0)},$$

where

$$\begin{aligned}
f(x) &= \lambda e^{-\lambda x},\ \ \lambda > 0, \\
P(X \leq x_0 + y, X \geq x_0) &= \int_{x_0}^{x_0+y} \lambda e^{-\lambda x} dx, \\
&= \frac{\lambda e^{-\lambda x}}{-\lambda}\bigg|_{x_0}^{x_0+y}, \\
&= e^{-\lambda y}. \\
P(X \geq x_0) &= \int_{x_0}^{\infty} \lambda e^{-\lambda x} dx, \\
&= \frac{\lambda e^{-\lambda x}}{-\lambda}\bigg|_{x_0}^{\infty}, \\
&= e^{-\lambda x_0}. \\
\therefore P(X \leq x_0 + y | X \geq x_0) &= \frac{e^{-\lambda y}}{e^{-\lambda x_0}}, \\
&= e^{-\lambda(y - x_0)}.
\end{aligned}$$

23. **Expected values**

(a)

$$\begin{aligned}
E(X) &= \sum_{\forall x} x f(x), \\
&= \bigg(\frac{1}{4} \times \frac{1}{4}\bigg) + \bigg(-2 \times \frac{1}{4}\bigg) + \bigg(\frac{-1}{2} \times 0\bigg) + \bigg(\frac{3}{7} \times \frac{3}{10}\bigg) + \bigg(\frac{8}{11} \times \frac{1}{5}\bigg), \\
\therefore E(X) &= -0.1635.
\end{aligned}$$

(b)

$$\begin{aligned}
f(x) &= \frac{\lambda^x e^{-\lambda}}{x!}, \\
E(X) &= \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!}, \\
&= \sum_{x=1}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!}, \\
&= \lambda \sum_{x=1}^{\infty} \frac{\lambda^{(x-1)} e^{-\lambda}}{(x-1)!}, \\
\therefore E(X) &= \lambda.
\end{aligned}$$

24. (a) TRUE if X is a discrete random variable which takes only one particular value with $E(X) = xf(x) = x$. FALSE if X is a continuous random variable, for example, continuous uniform distribution with $a = 0$ and $b = 1$.

(b) FALSE. Since $E(X) = \frac{a+b}{2}$ and $-\infty < a < b < \infty$, it depends on both the interval length and signs of a and b.

(c) TRUE. This implies that the random variable X is discrete and hence $E(X) = \sum_{\forall x} xf(x)$.

(d) TRUE. Since $E(X) = \lambda$ and $\lambda \in (0, \infty)$ so the expected value always exists since λ always exists.