Exercises for the lecture

Mathematics III

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Solutions to Exercise 07

Tasks

25. **Gaussian distribution**

(a) Let X be a random variable that denotes the score in an exam, then $X \sim N(527, 12544)$. We need,

$$\begin{aligned} P(X > 500) &= P\left(Z > \frac{500 - 527}{\sqrt{12544}}\right), \\ &= P(Z > -0.2410), \\ &= 1 - \phi(-0.2411), \\ &= 1 - 0.40517, \\ &= 0.59483. \end{aligned}$$

(b) We need to find a such that

$$P(Z > a) = 0.05,$$

where

$$\begin{aligned} a &= \frac{x - 527}{\sqrt{12544}}, \\ \therefore P(Z \le a) &= 0.95, \\ \phi(a) &= 0.95, \\ a &= \phi^{-1}(0.95), \\ a &= 1.65, \\ \therefore x &= 711.80. \end{aligned}$$

26. **Exponential distribution**
Let X be the time between buying a plane ticket and departure, then $X \sim Exp(\lambda)$ and $E(X) = \frac{1}{\lambda} = 15$. We need to find,

$$\begin{aligned} P(X < 10) &= \int_0^{10} \lambda e^{-\lambda x} dx, \\ &= -e^{\lambda x}\Big|_0^{10}, \\ &= 1 - e^{-10/15}, \\ &= 0.4866. \end{aligned}$$

27. **Calculation of expected values**

(a) Let X be the number of trials until the first successful trial, including the success. Then X has the First Success distribution with parameter θ; we denote this by $X \sim FS(\theta)$.

$$f(X) = (1-\theta)^{X-1}\theta.$$

$$L(\theta) = \theta^n(1-\theta)^{\sum_{i=1}^n X_i - n},$$

$$lnL(\theta) = nln\theta + (\sum_{i=1}^{n} X_i - n)ln(1-\theta),$$

$$\frac{dlnL(\theta)}{d\theta} = \frac{n}{\theta} - \frac{\sum_{i=1}^n X_i - n}{(1-\theta)} = 0,$$

$$\therefore \widehat{\theta} = \frac{1}{\bar{X}}.$$

(b) Let $Y = g(x) = 2^X$ where Y is the number of earnings received, for example, if heads appeared for the first time on the fourth toss, he/she collects 16 cents, it is to his/her advantage to get a very long run of tails before heads appears. With probability one, heads will eventually appear and he/she will collect some money.

(c) Let

$$P\{X = i\} = \left(\frac{1}{2}\right)^i.$$

For example,

$$P\{X = 1\} = \left(\frac{1}{2}\right)^1,$$

$$P\{X = 2\} = \left(\frac{1}{2}\right)^2,$$

etc.

$$\begin{aligned}\therefore E(Y) &= 2P\{Y=2\} + 4P\{Y=4\} + 8P\{Y=8\} + \dots \\ &= \left(2 * \frac{1}{2}\right) + \left(4 * \frac{1}{4}\right) + \left(8 * \frac{1}{8}\right) + \dots. \\ &= \infty.\end{aligned}$$

(d) On the other hand, $X \sim FS(1/2)$ and therefore $E(X) = 2$. Thus $E(2^X) = \infty$ while $2^{E(X)} = 4$.

28. **Multiple select task**

(a) FALSE. For example, the normal distribution is unbounded but the expected value still exists.

(b) FALSE. Since X is discrete.

(c) TRUE. For example, the normal distribution with $E(X) = 0$ is symmetric at $x = 0$.

(d) TRUE. Since $E(2X) = 2E(X) = 0$.