Exercises for the lecture

Mathematics III

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Solutions to Exercise 08

Tasks

29. **Calculation of variances.**

(a)

$$
\begin{aligned}
E(X) &= \sum_{\forall x} x f(x), \\
&= \left(-2 * \frac{1}{4}\right) + \left(\frac{-1}{2} * \frac{1}{4}\right) + \left(\frac{3}{7} * \frac{3}{10}\right) + \left(\frac{8}{11} * \frac{1}{5}\right), \\
&= -0.350974. \\
E(X^2) &= \sum_{\forall x} x^2 f(x), \\
&= \left(4 * \frac{1}{4}\right) + \left(\frac{1}{4} * \frac{1}{4}\right) + \left(\frac{9}{49} * \frac{3}{10}\right) + \left(\frac{64}{121} * \frac{1}{5}\right), \\
&= 1.223387. \\
Var(X) &= E(X^2) - (E(X))^2, \\
\therefore Var(X) &= 1.100204.
\end{aligned}
$$

(b)

$$Var(X) = E(X^2) - (E(X))^2,$$

$$f(x) = \lambda e^{-\lambda x},$$

$$E(X) = \int_0^\infty x \lambda e^{-\lambda x} dx,$$

$$= \lambda \left[\int_0^\infty x e^{-\lambda x} dx \right],$$

Let

$$
\begin{aligned}
u = x, &\implies du = dx, \\
dv = e^{-\lambda x} &\implies v = e^{-\lambda x} / -\lambda,
\end{aligned}
$$

$$= \lambda \left[\frac{x e^{-\lambda x}}{-\lambda} + \int_0^\infty \frac{e^{-\lambda x}}{\lambda} dx \right],$$

$$= \lambda \left[\frac{x e^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \right]_0^\infty,$$

$$E(X) = \frac{1}{\lambda}.$$

$$E(X^2) = \int_0^\infty x^2 \lambda e^{-\lambda x} dx,$$
$$= \lambda \left[\int_0^\infty x^2 e^{-\lambda x} dx \right],$$

Let

$$u = x^2, \implies du = 2x dx,$$
$$dv = e^{-\lambda x} \implies v = e^{-\lambda x} / -\lambda,$$

$$= \lambda \left[\frac{x^2 e^{-\lambda x}}{-\lambda} + \frac{2}{\lambda} \int_0^\infty x \frac{e^{-\lambda x}}{\lambda} dx \right],$$
$$= \lambda \left[\frac{x^2 e^{-\lambda x}}{-\lambda} + \frac{2}{\lambda^3} \right]_0^\infty,$$
$$E(X^2) = \frac{1}{\lambda^2}.$$

$$\therefore Var(X) = \frac{1}{\lambda}.$$

30. **Calculation of medians.**

(a) We need $P(X \leq m) = 0.5$ where m is the median, this is given by $m = -1/2$ from the interval $-1/2 \leq x < 3/7$.

(b) In this case, the median m is given by,

$$m = \begin{cases} 0 & p < 1/2 \\ [0, 1] & p = 1/2 \\ 1 & p > 1/2 \end{cases}.$$

31. **Covariance and correlation.**

(a)

$$Cov(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2),$$
$$E(Y_1) = E(X_1) + 2E(X_2),$$
$$= 0.$$
$$E(Y_2) = 4E(X_1) - 3E(X_2),$$
$$= 0.$$

$$E(Y_1 Y_2) = 4E(X_1^2) - 3E(X_1 X_2) + 8E(X_1 X_2) - 6E(X_2^2),$$
$$E(X_1^2) = \sigma,$$
$$E(X_2^2) = \sigma,$$
$$E(X_1 X_2) = E(X_1)E(X_2),$$
$$= 0.$$
$$\therefore E(Y_1 Y_2) = -2\sigma.$$

(b)

$$
\begin{aligned}
Corr(Y_1Y_2) &= \frac{Cov(Y_1Y_2)}{\sqrt{Var(Y_1)Var(Y_2}}, \\
Var(Y_1) &= E(Y_1^2), \\
&= E(X_1^2 + 4X_1X_2 + 4X_2^2), \\
&= 5\sigma. \\
Var(Y_2) &= E(Y_2^2), \\
&= E(16X_1^2 - 24X_1X_2 + 9X_2^2), \\
&= 25\sigma. \\
\therefore Corr(Y_1Y_2) &= \frac{-2\sigma}{\sqrt{(5\sigma)(25\sigma)}}, \\
&= \frac{-2}{5\sqrt{5}}.
\end{aligned}
$$

32. **Multiple select task.**

(a) FALSE. The median always exists while the mean does not always exist, for example, the cauchy distribution.

(b) FALSE. For example, with $P(X = 1) = 1/3$, $P(X = 2) = 1/3$, and $P(X = -3) = 1/3$, $E(X) = 0$ but $x_m = 2$, so this is not the case in general.

(c) TRUE. For a symmetric distribution, mean=mode=median.

(d) FALSE. For example, for $P(X = 1) = 1/4$, $P(X = 2) = 1/8$, and $P(X = 3) = 5/8$, then the median is clearly determined as $x_m = 3$.